Some Elementary Tiling Theory

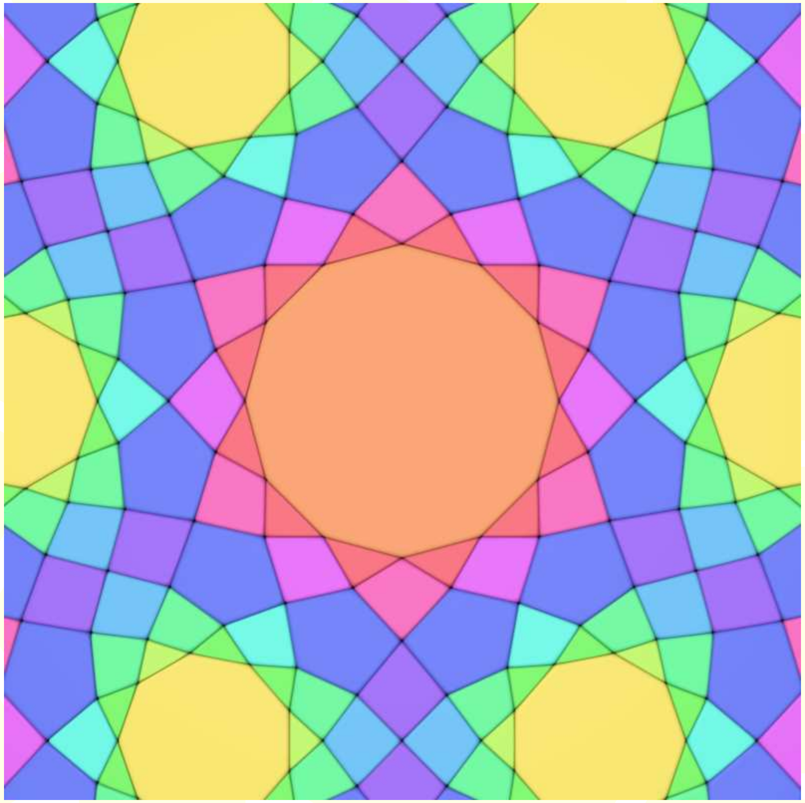
Santa Barbara, August 2008

Olaf Delgado-Friedrichs

The Australian National University - Supercomputer Facility

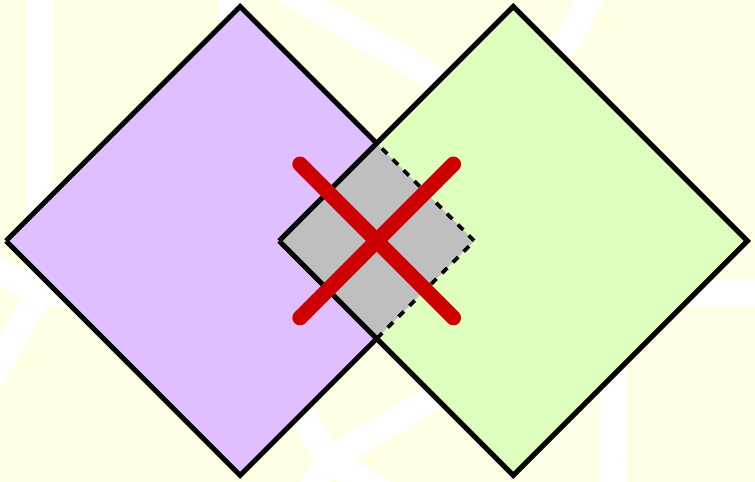
What is a tiling?

- Partition of a manifold (e.g. the plane).



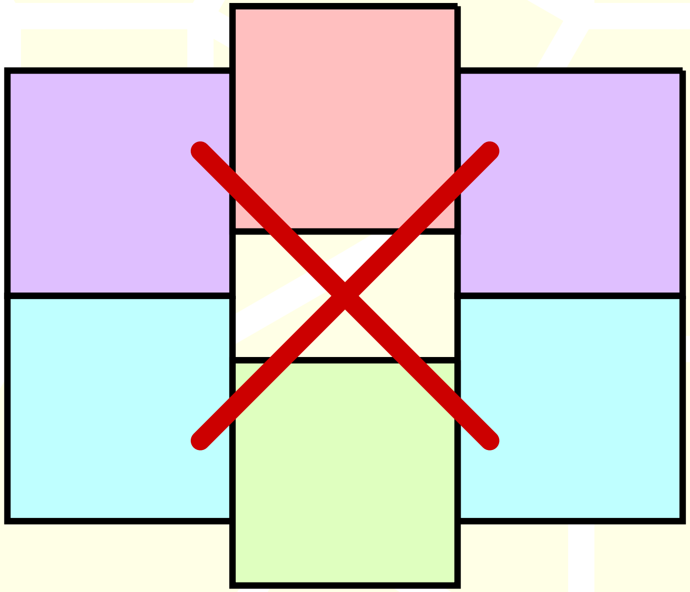
What is a tiling?

- Partition of a manifold (e.g. the plane).
- No overlaps.



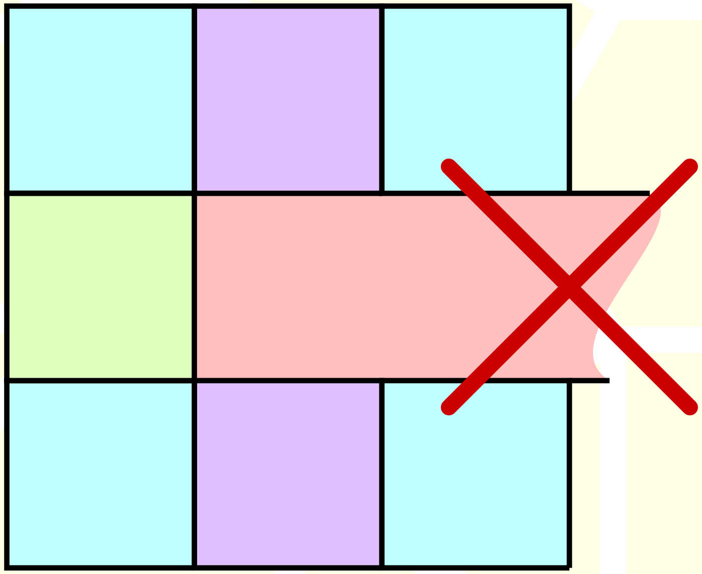
What is a tiling?

- Partition of a manifold (e.g. the plane).
- No overlaps.
- No holes.



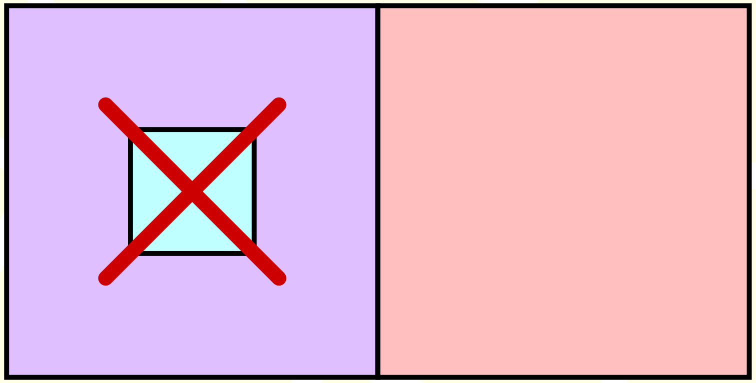
What is a tiling?

- Partition of a manifold (e.g. the plane).
- No overlaps.
- No holes.
- Tiles are bounded.



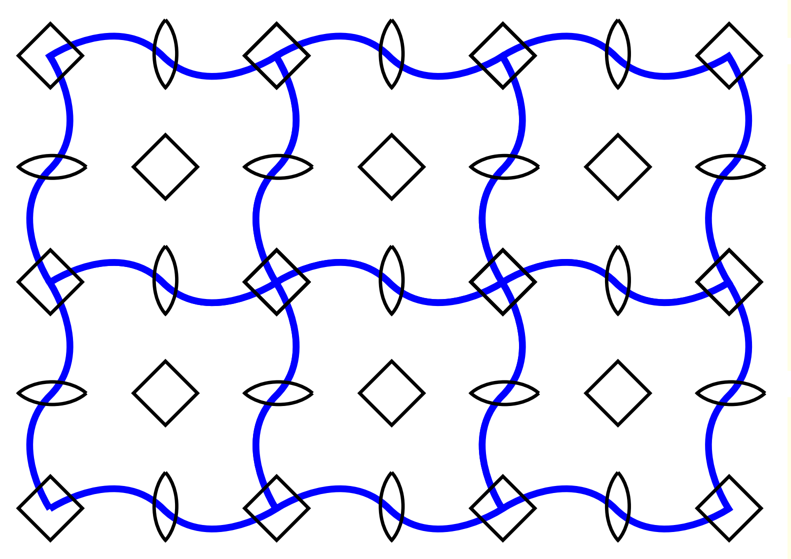
What is a tiling?

- Partition of a manifold (e.g. the plane).
- No overlaps.
- No holes.
- Tiles are bounded.
- Tiles are cells (have no holes).



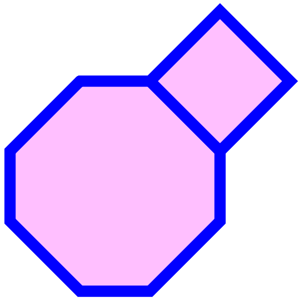
Symmetries

- Equivariant tilings have specified symmetries.



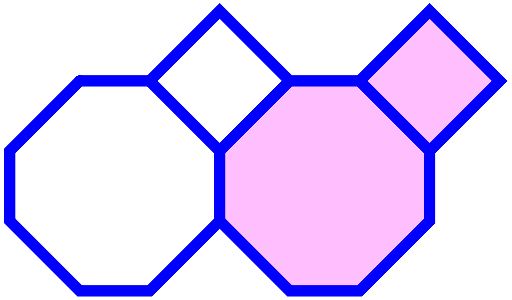
Symmetries

- Equivariant tilings have specified symmetries.
- A periodic tiling consists of translated copies of a compact motif (containing finitely many tiles).



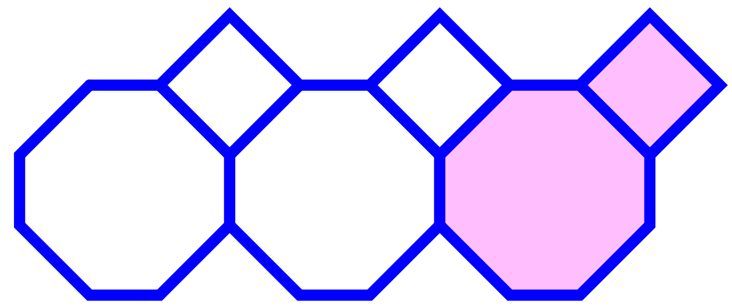
Symmetries

- Equivariant tilings have specified symmetries.
- A periodic tiling consists of translated copies of a compact motif (containing finitely many tiles).



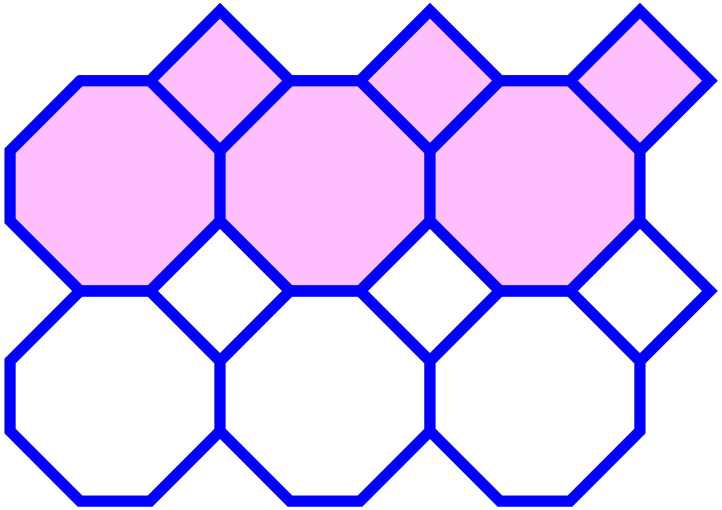
Symmetries

- Equivariant tilings have specified symmetries.
- A periodic tiling consists of translated copies of a compact motif (containing finitely many tiles).



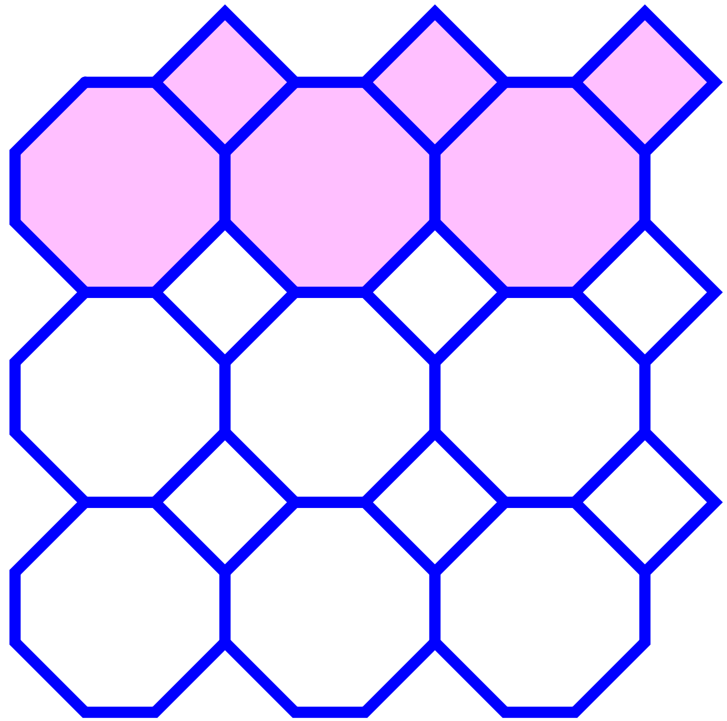
Symmetries

- Equivariant tilings have specified symmetries.
- A periodic tiling consists of translated copies of a compact motif (containing finitely many tiles).



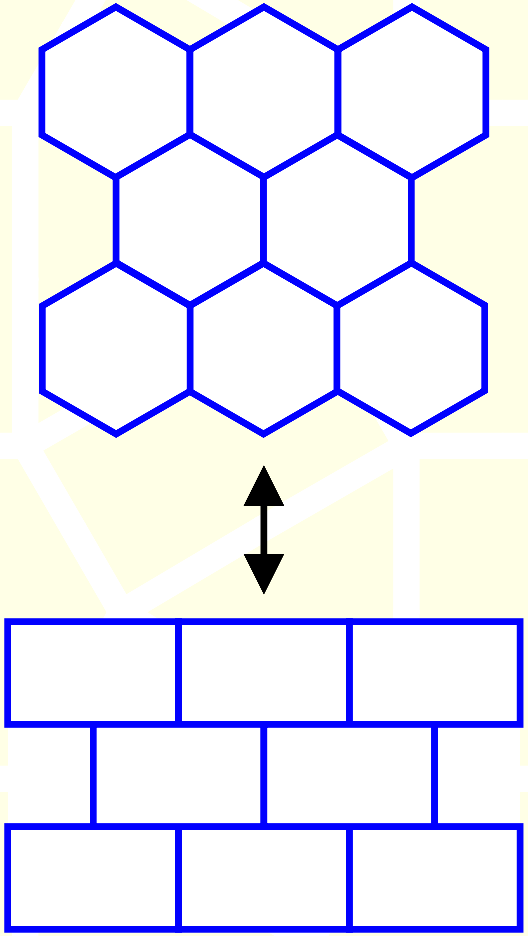
Symmetries

- Equivariant tilings have specified symmetries.
- A periodic tiling consists of translated copies of a compact motif (containing finitely many tiles).



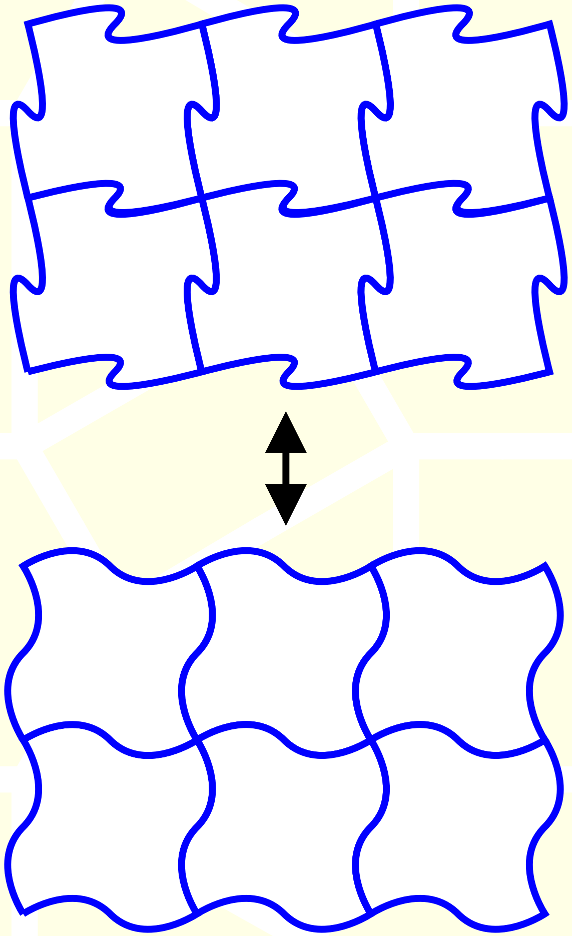
Equivalence

- Topologically equivalent tilings can be deformed into each other.



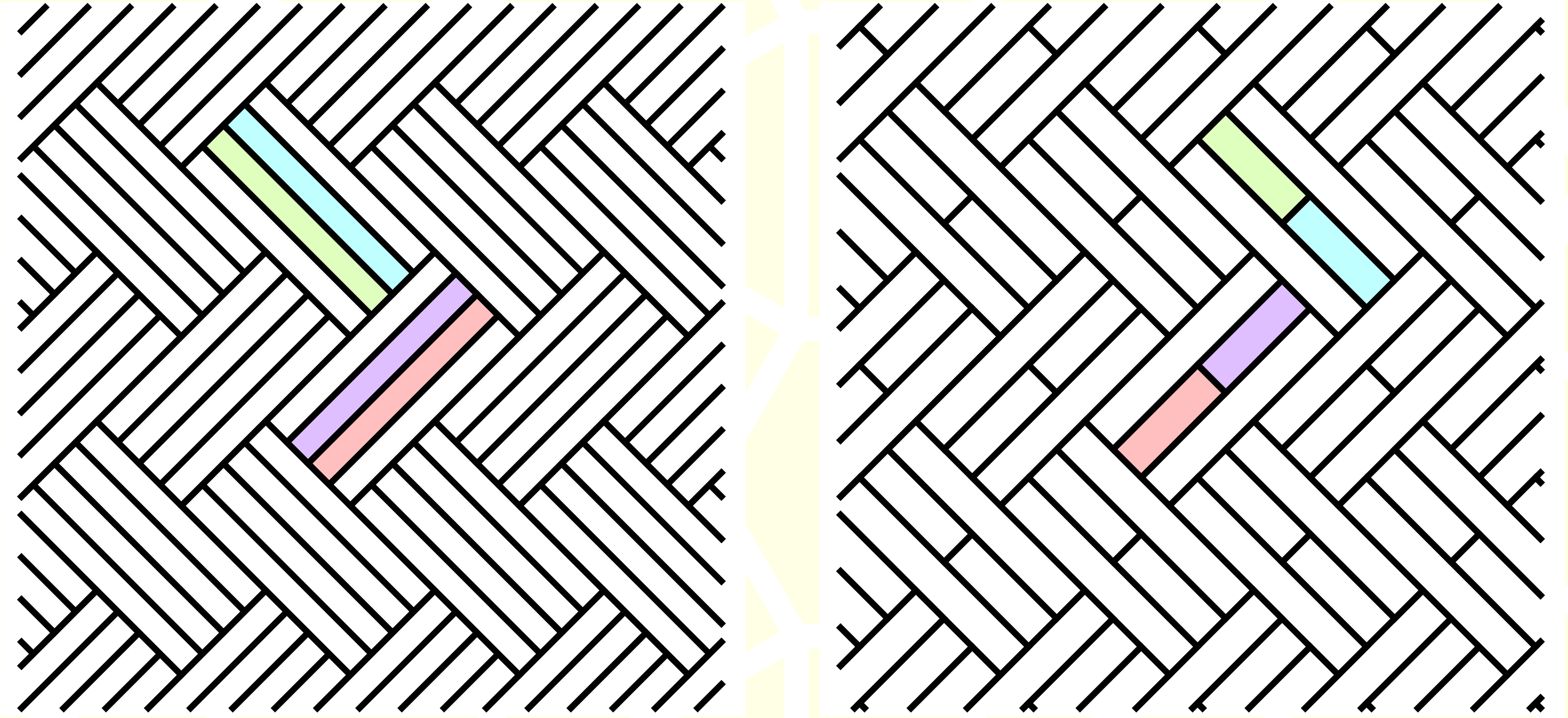
Equivalence

- Topologically equivalent tilings can be deformed into each other.
- In equivariantly equivalent tilings, the deformation respects symmetries.



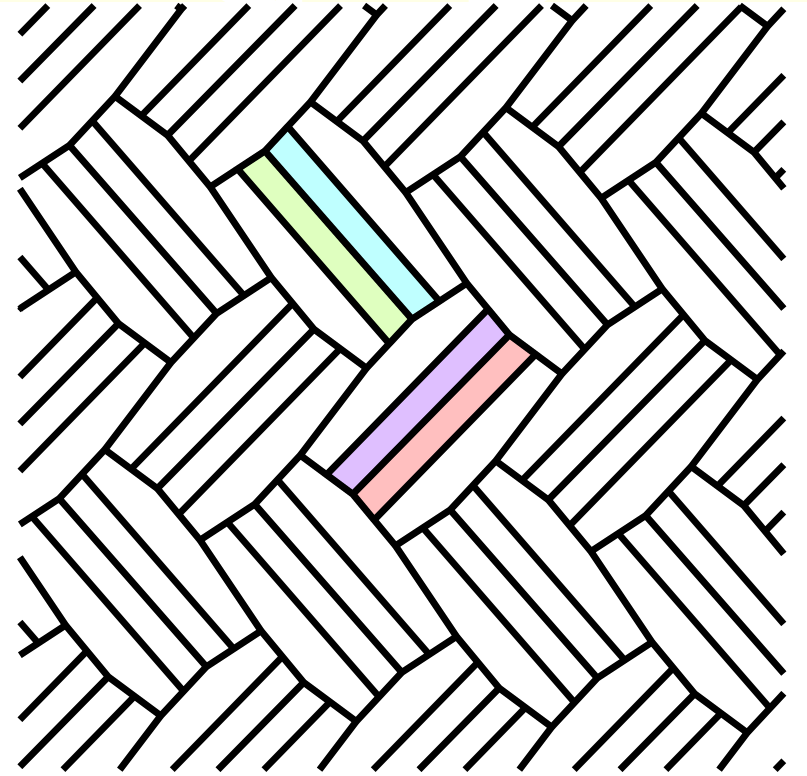
Are these the same?

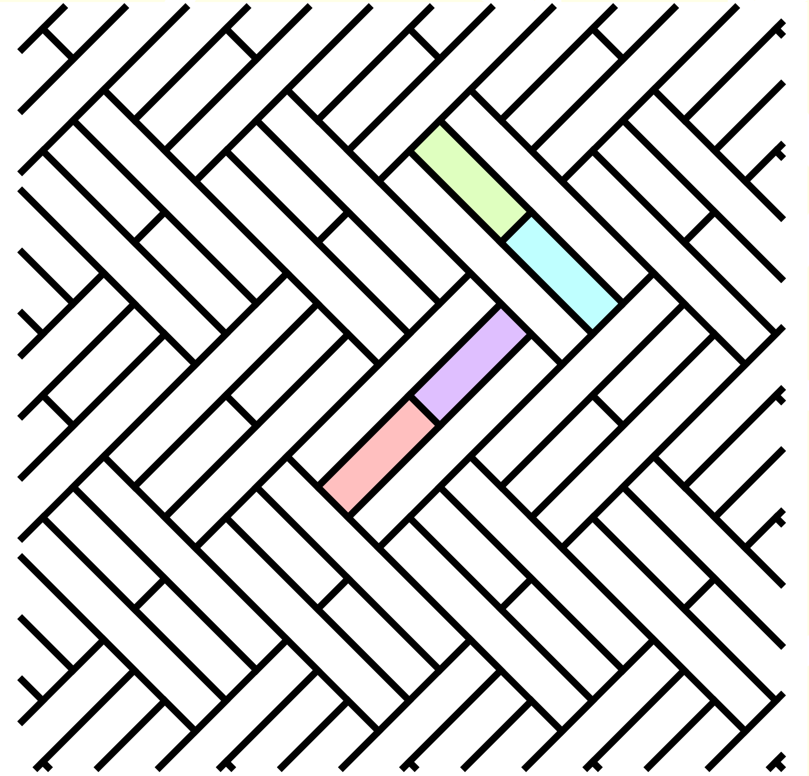
A problem posed by LOTHAR COLLATZ (1910–1990).



Are these the same?

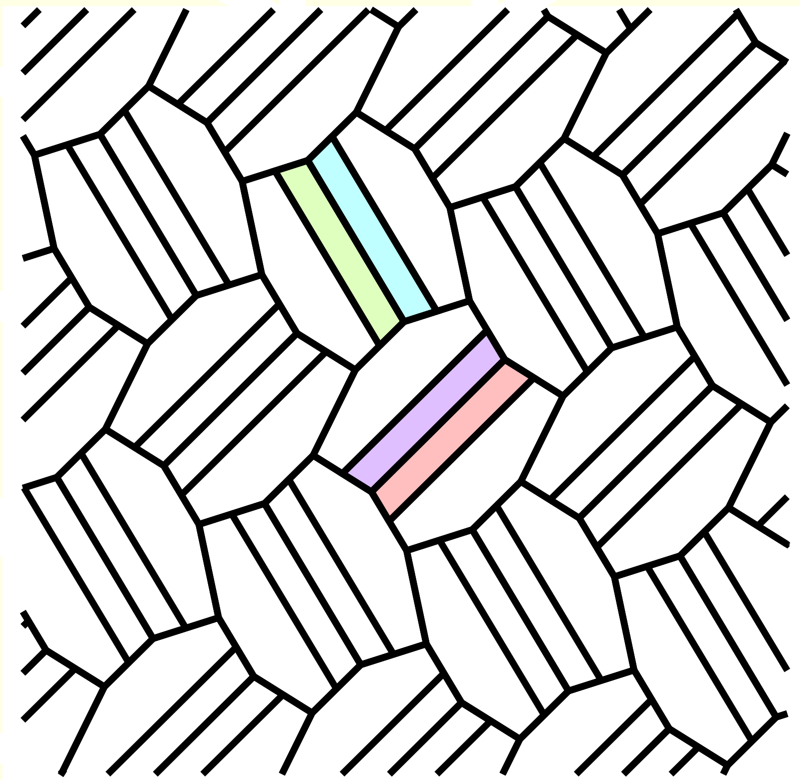
A problem posed by LOTHAR COLLATZ (1910–1990).

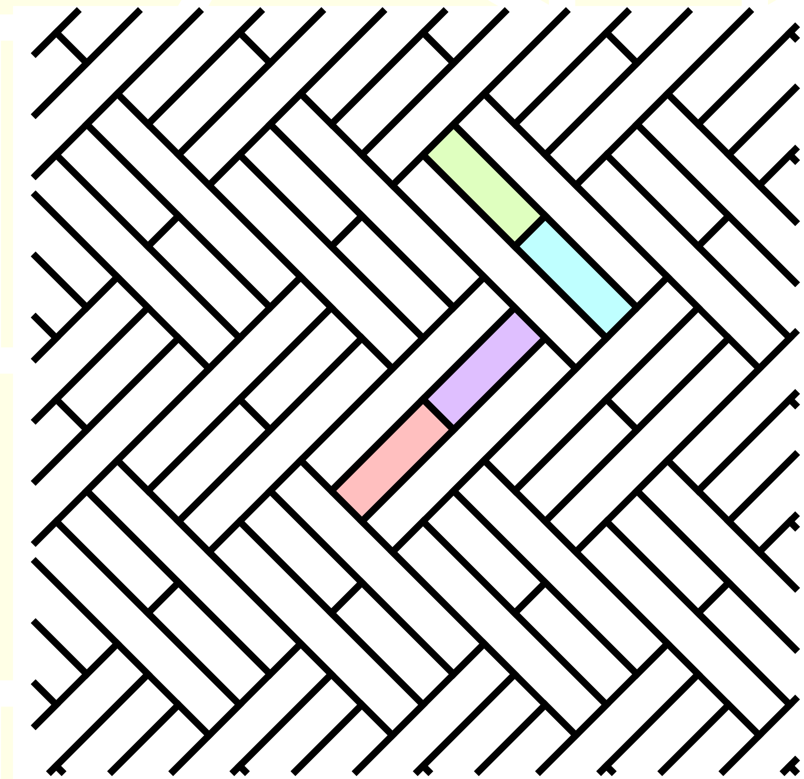




Are these the same?

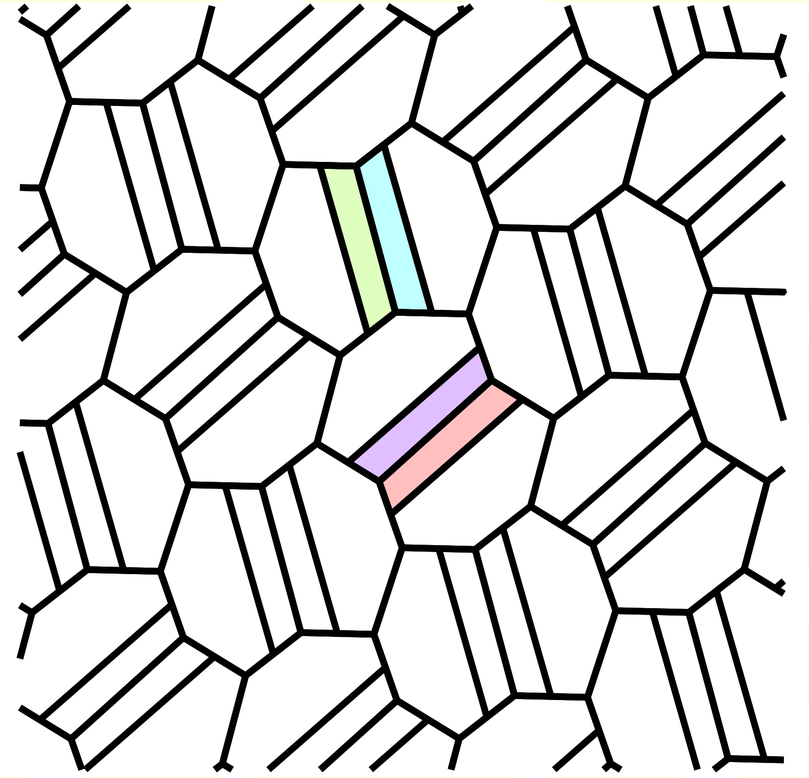
A problem posed by LOTHAR COLLATZ (1910–1990).

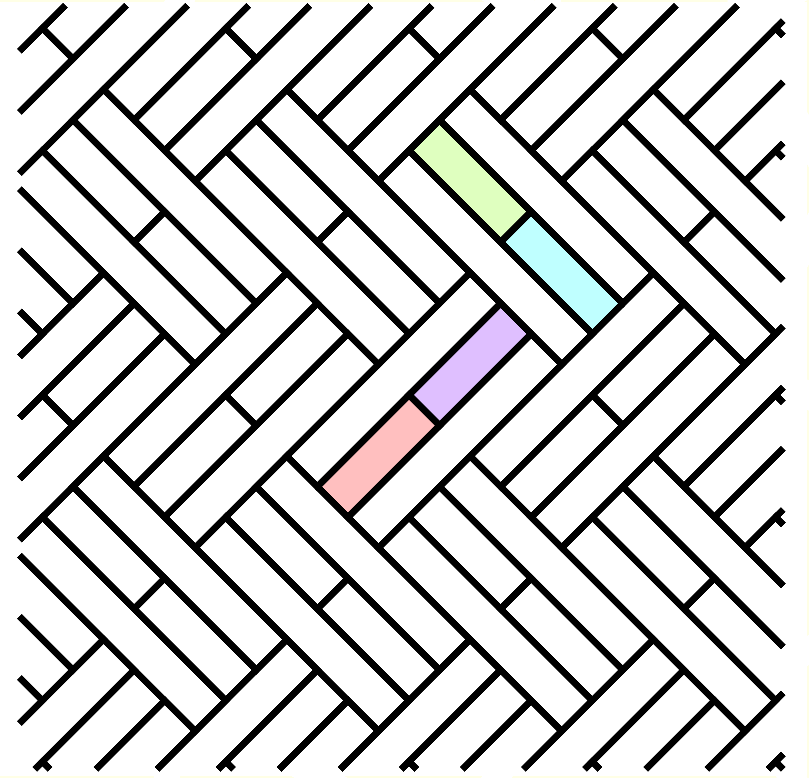




Are these the same?

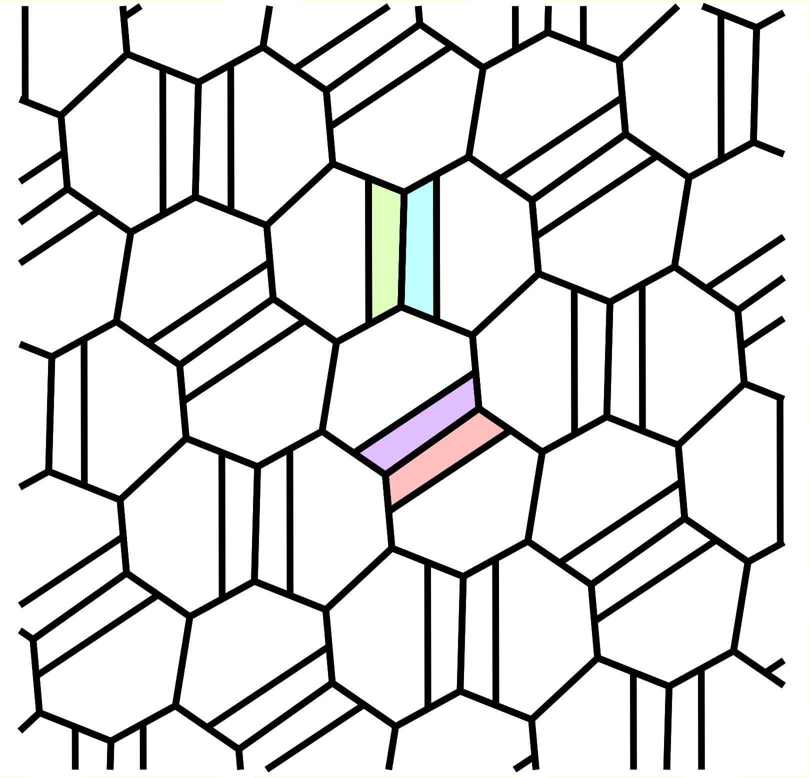
A problem posed by LOTHAR COLLATZ (1910–1990).

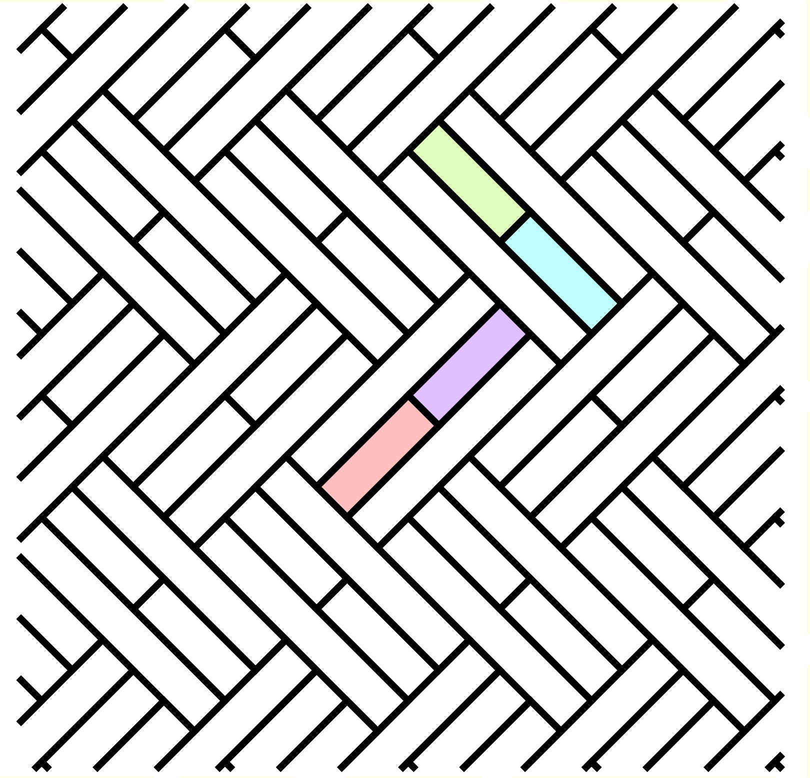




Are these the same?

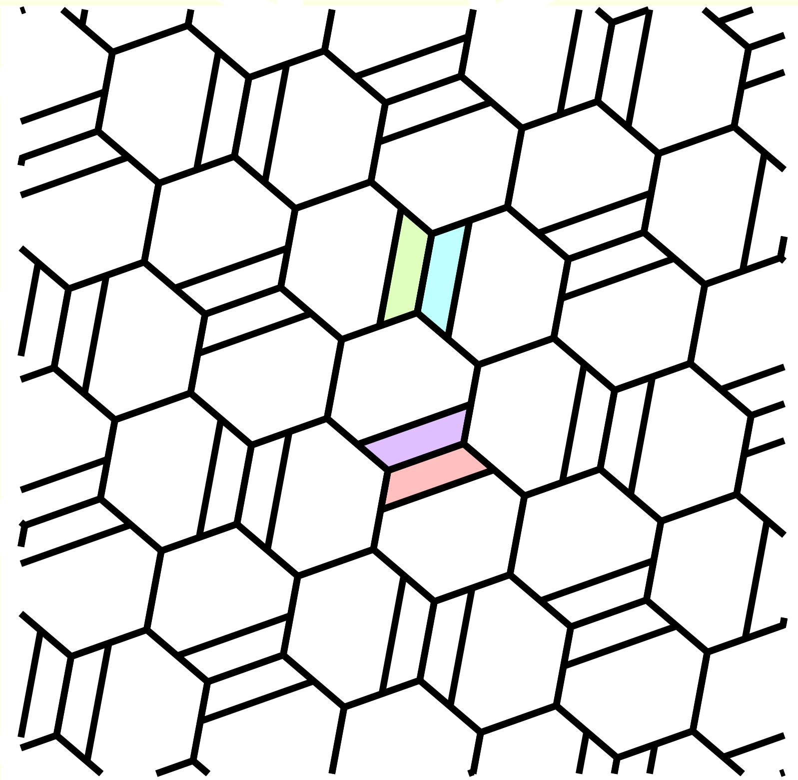
A problem posed by LOTHAR COLLATZ (1910–1990).

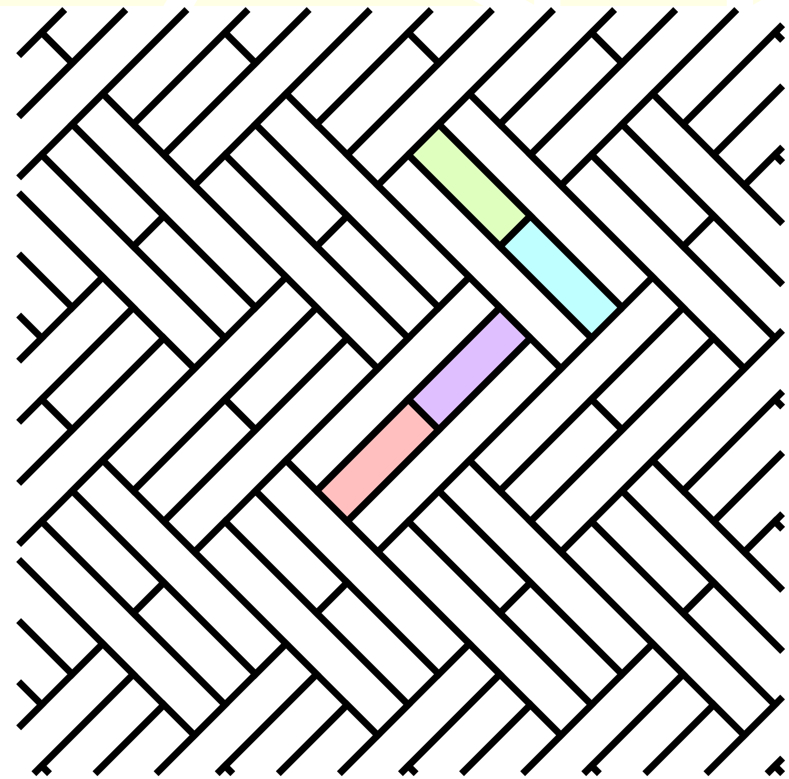




Are these the same?

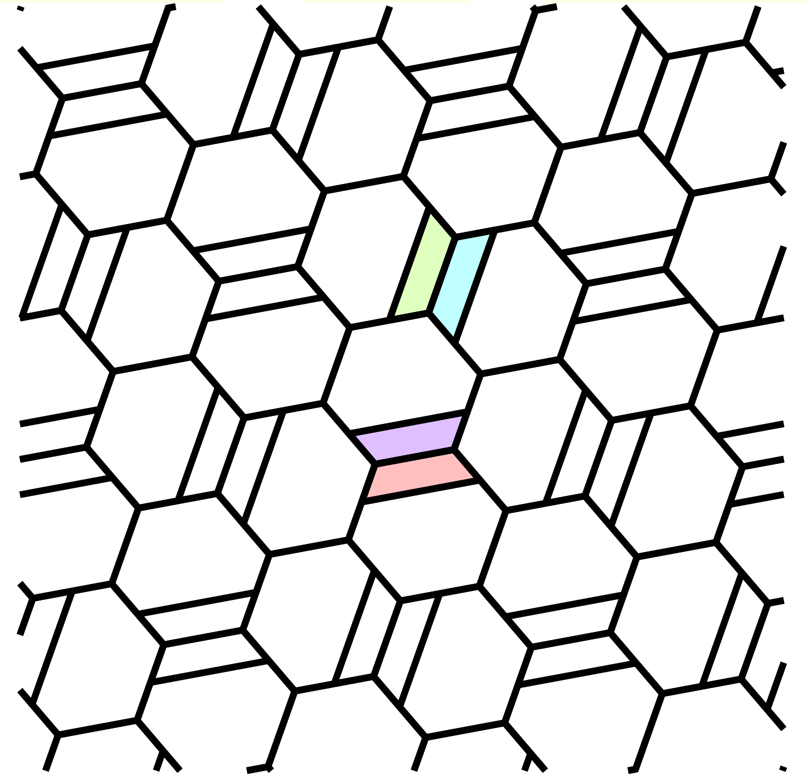
A problem posed by LOTHAR COLLATZ (1910–1990).

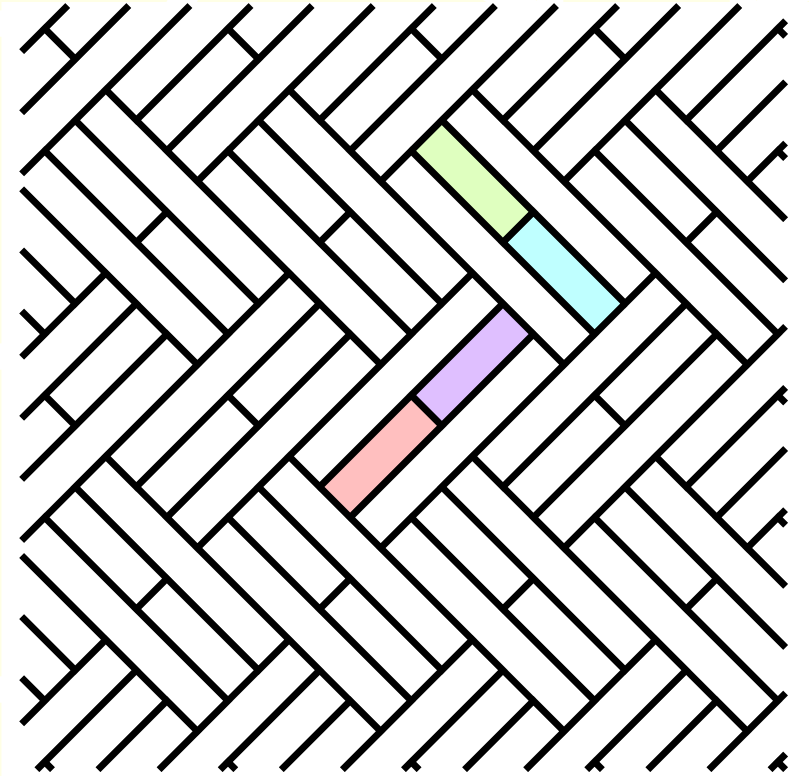




Are these the same?

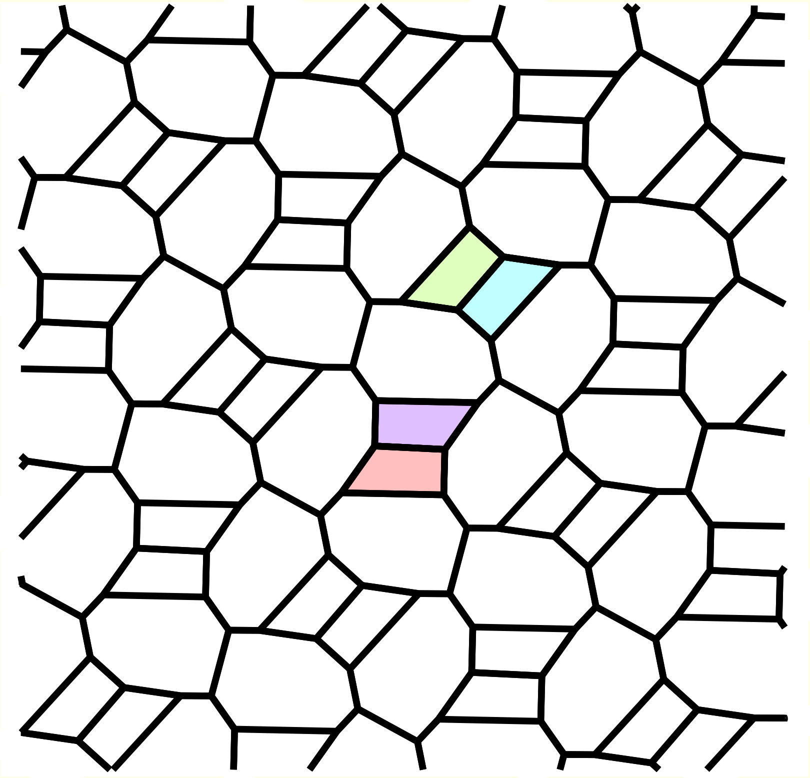
A problem posed by LOTHAR COLLATZ (1910–1990).

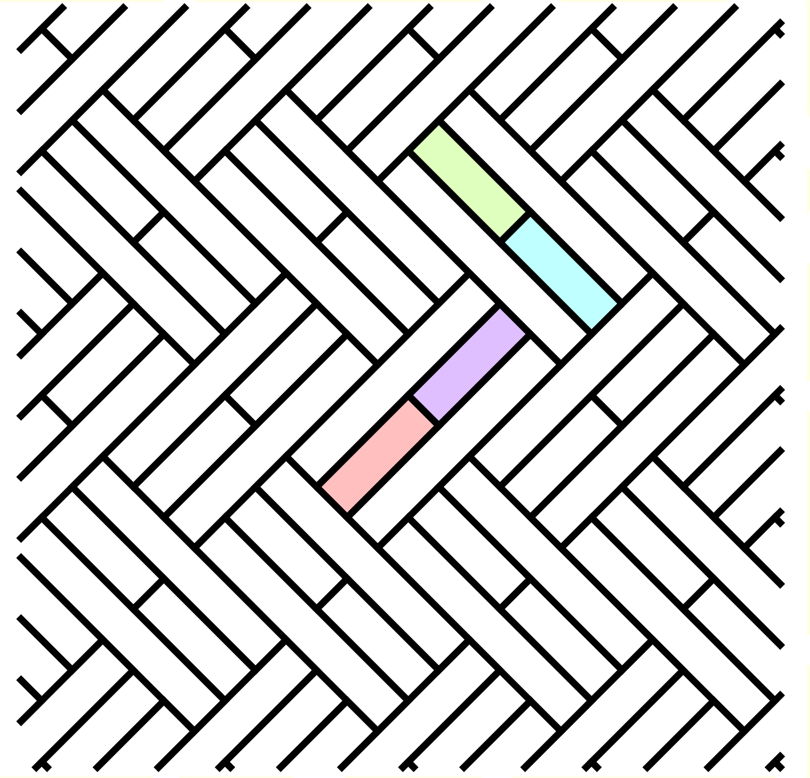




Are these the same?

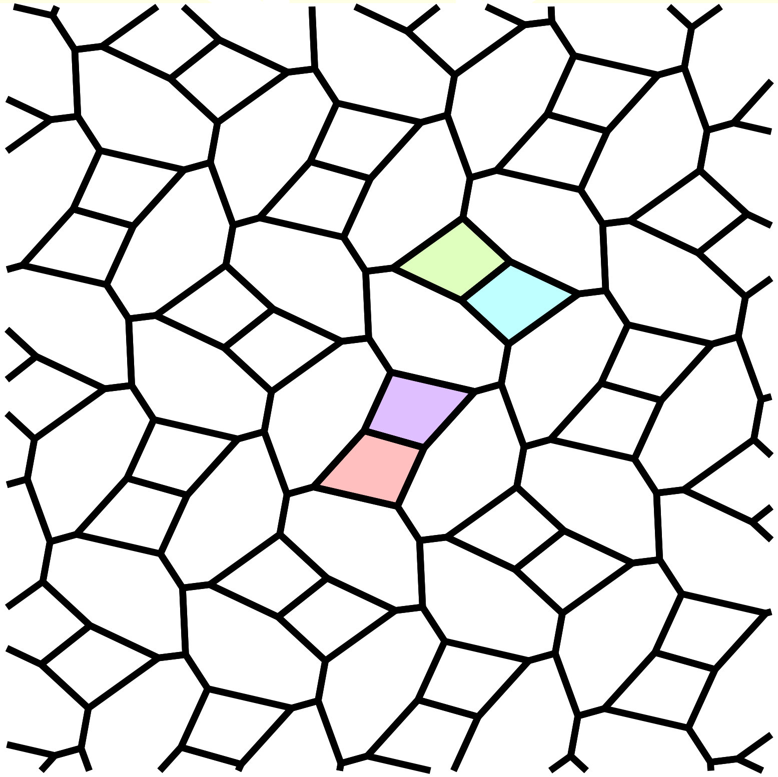
A problem posed by LOTHAR COLLATZ (1910–1990).

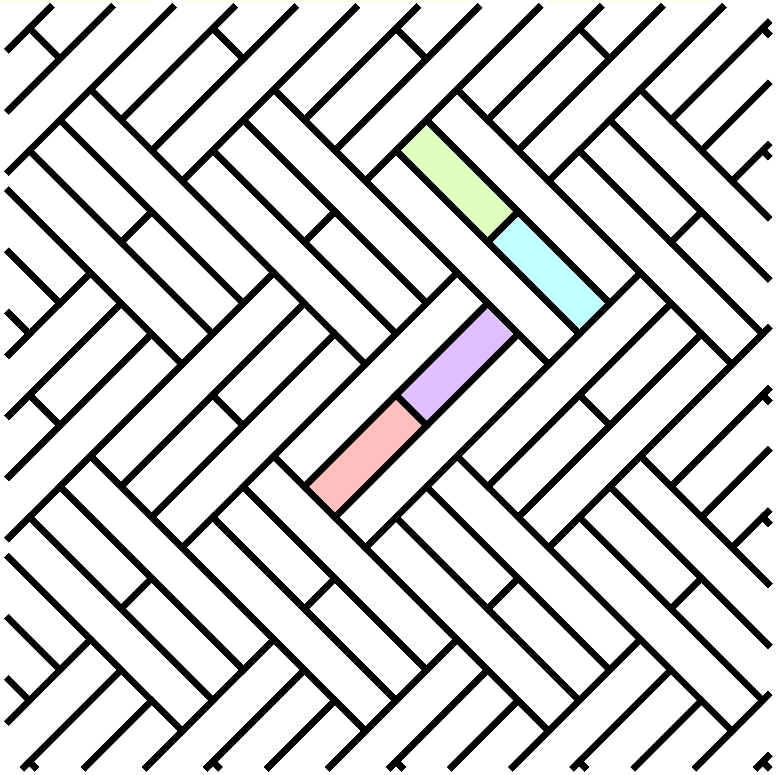




Are these the same?

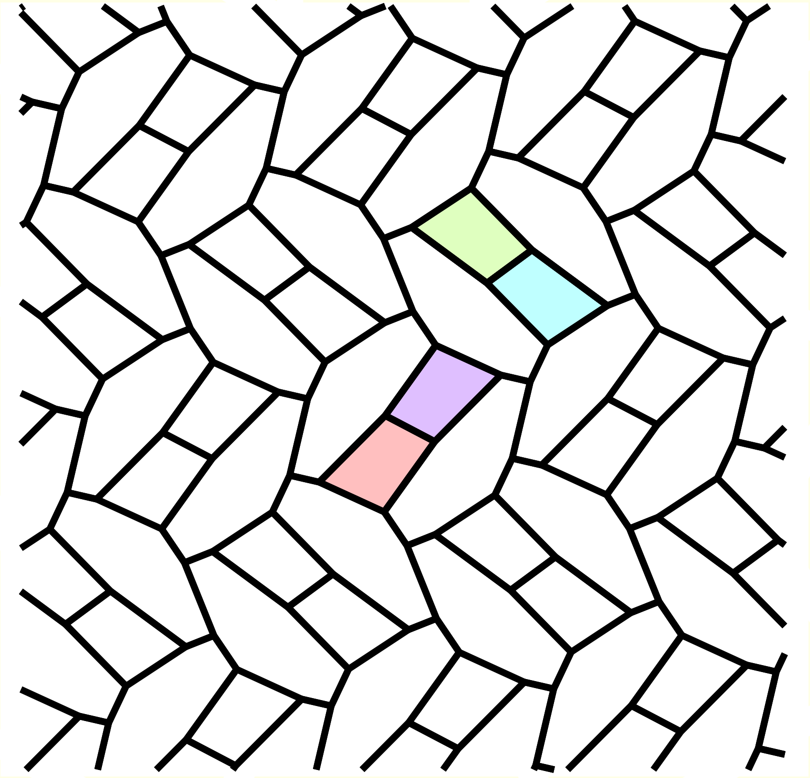
A problem posed by LOTHAR COLLATZ (1910–1990).

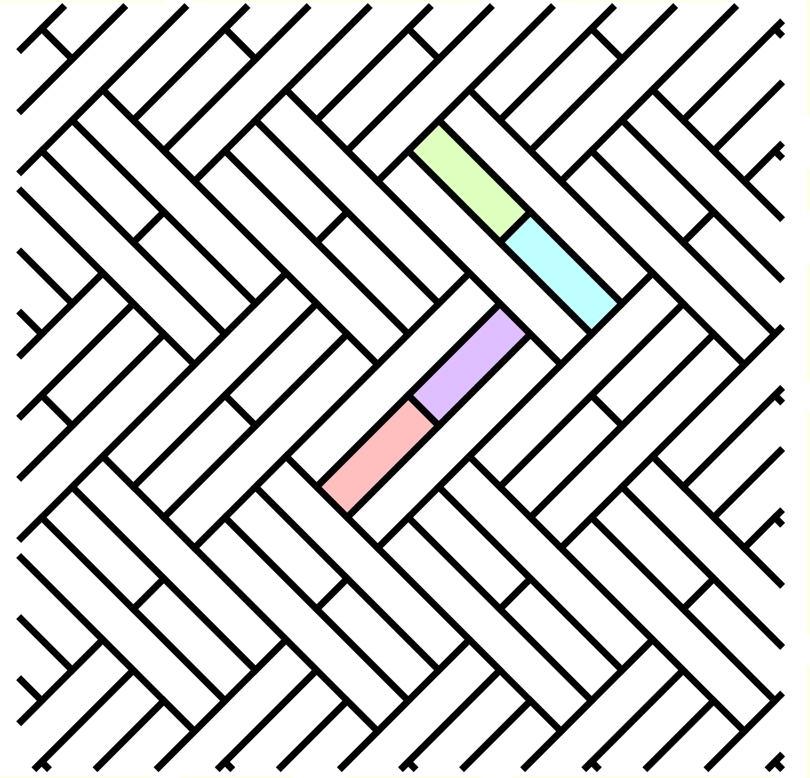




Are these the same?

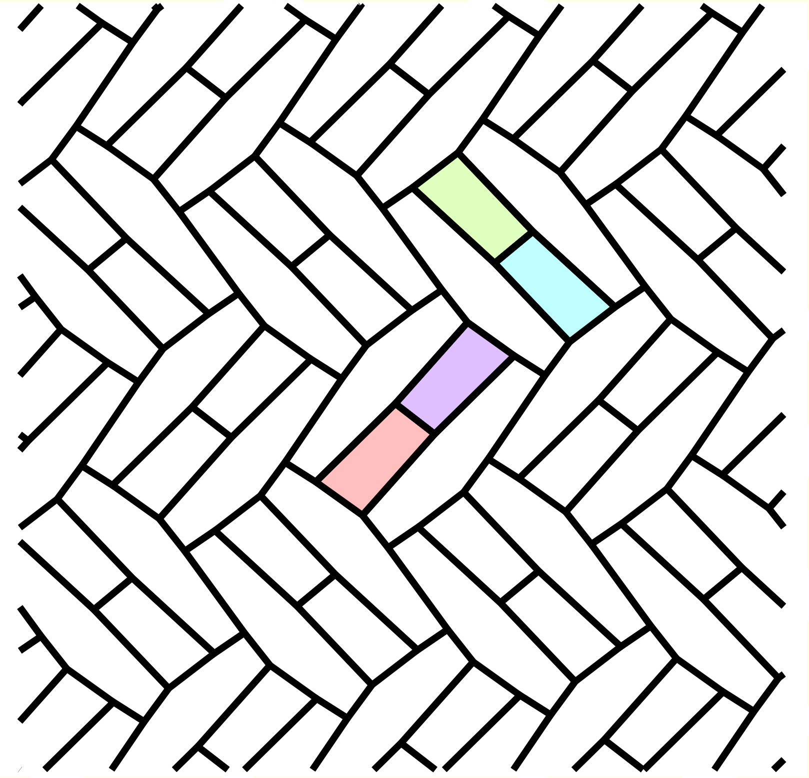
A problem posed by LOTHAR COLLATZ (1910–1990).

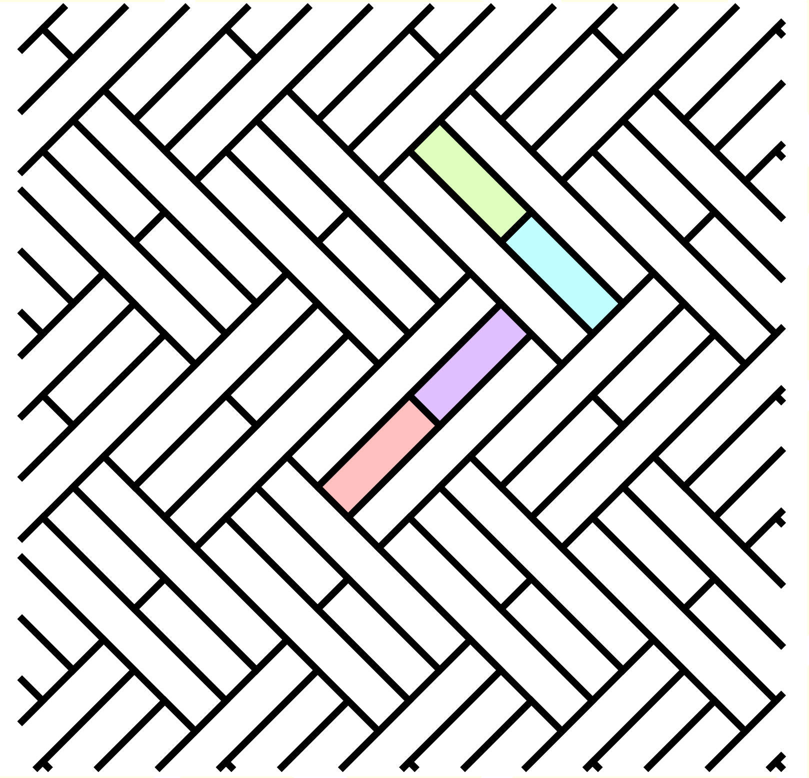




Are these the same?

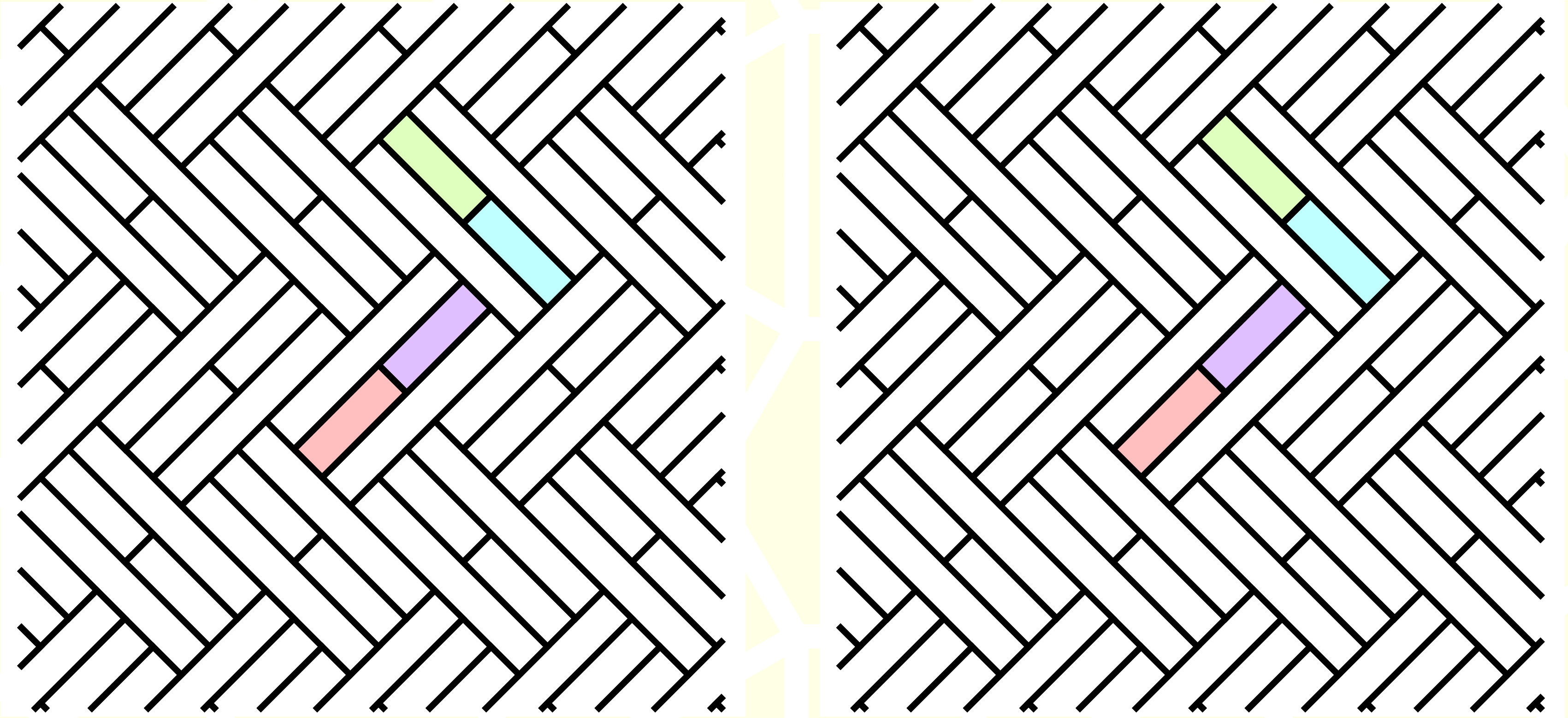
A problem posed by LOTHAR COLLATZ (1910–1990).





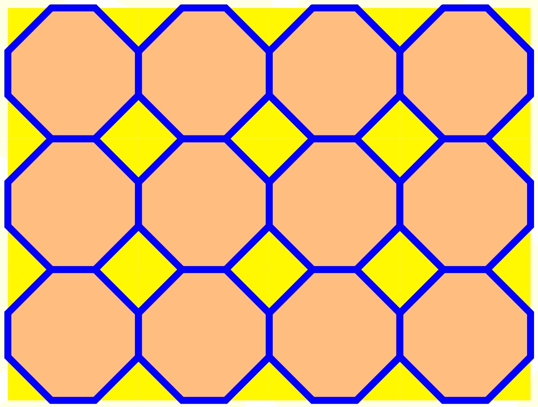
Are these the same?

A problem posed by Lothar Collatz (1910–1990).



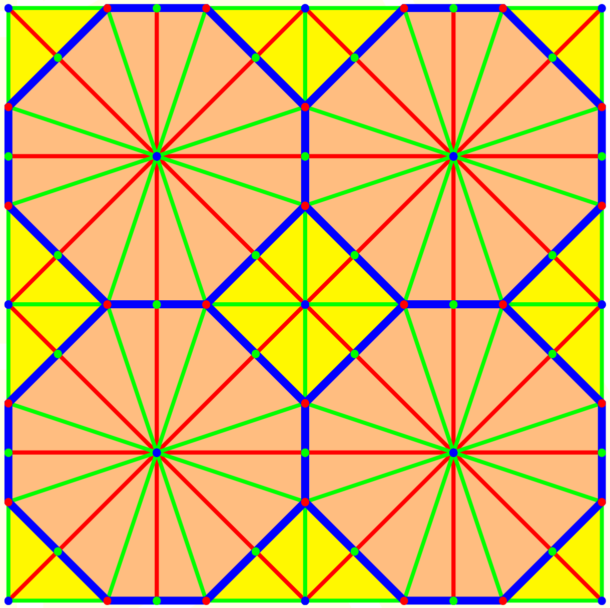
Yes, they are!

Barycentric triangulation

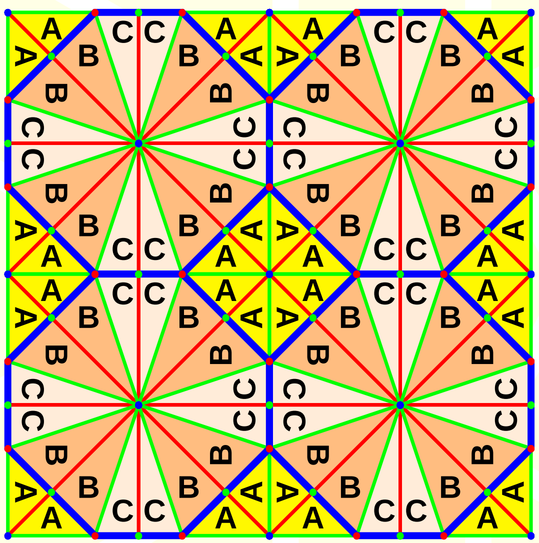


In order to represent tilings in a finite way, we start by dissecting tiles into triangles as shown below.

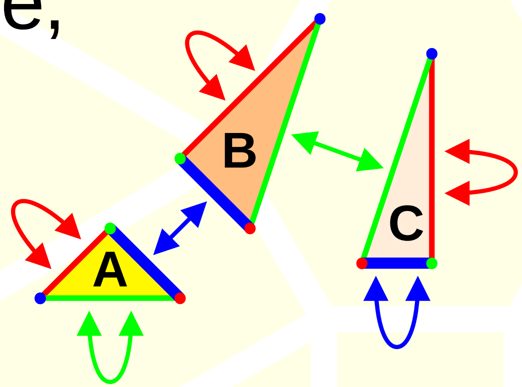
A color-coding later helps with the reassembly. Each corner receives the same color as the opposite side.



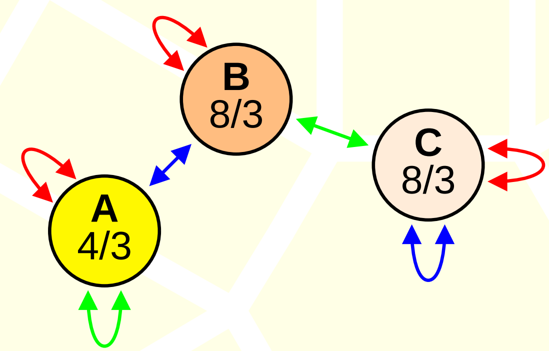
Symbols for tilings



Symmetric pieces get a common name, leading to compact assembly instructions.

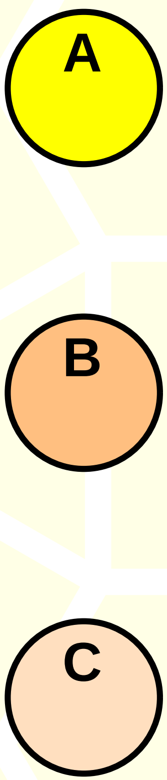


Face and vertex degrees replace particular shapes. The result is called a Delaney-Dress symbol (or shorter, a D-symbol.)



Ingredients

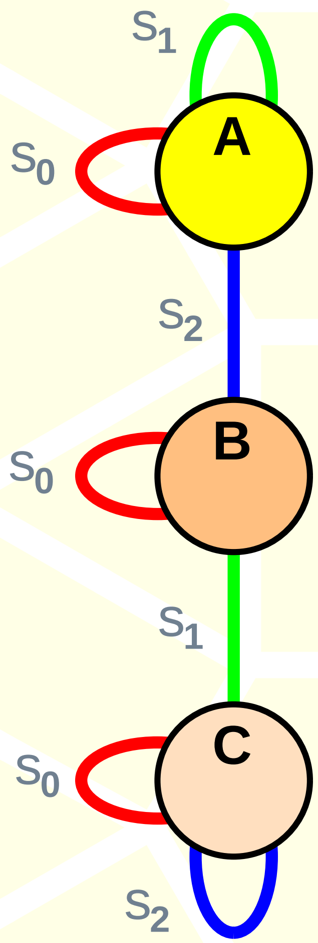
A D-symbol of dimension d consists of



- A finite set of so-called chambers,

Ingredients

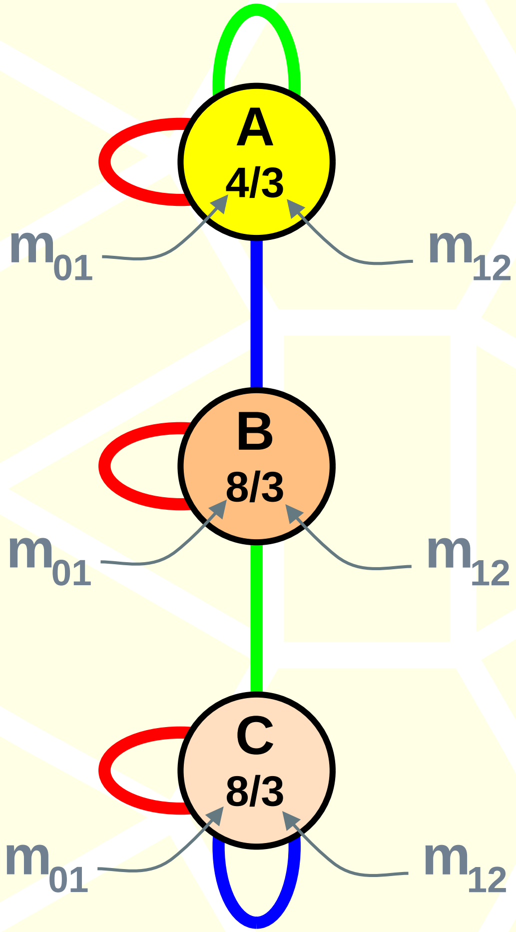
A D-symbol of dimension d consists of



- A finite set of so-called chambers,
- Operations $s_0, \ldots, s_d$ that map chambers to chambers,

Ingredients

A D-symbol of dimension d consists of

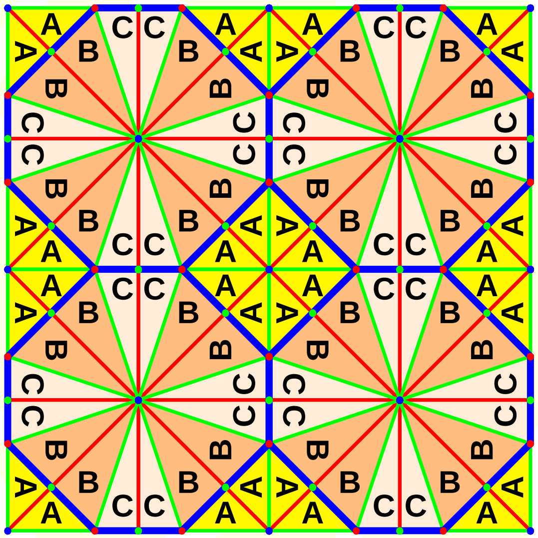


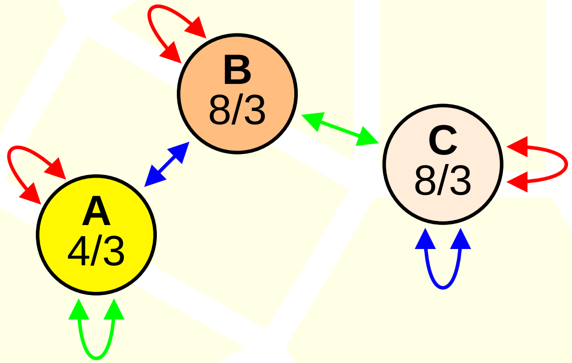
- A finite set of so-called chambers,
- Operations $s_0, \ldots, s_d$ that map chambers to chambers,
- Functions $m_{0,1}, \ldots, m_{d-1,d}$ that map chambers to integers.

Formal conditions

For C a chamber and $i, j \in \{0, \ldots, d\}$, we always have

- $s_i(s_i(C)) = C$

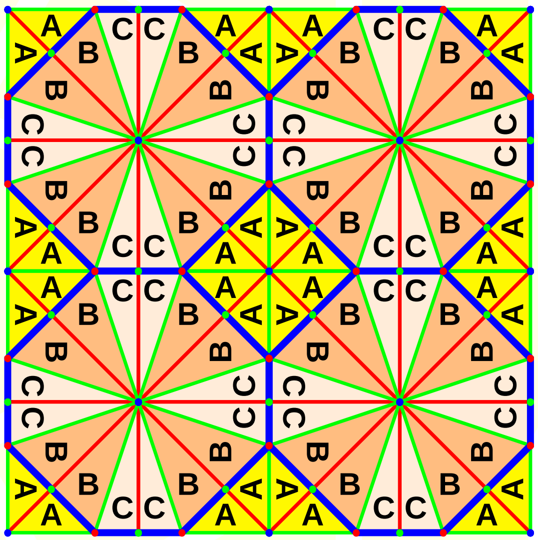


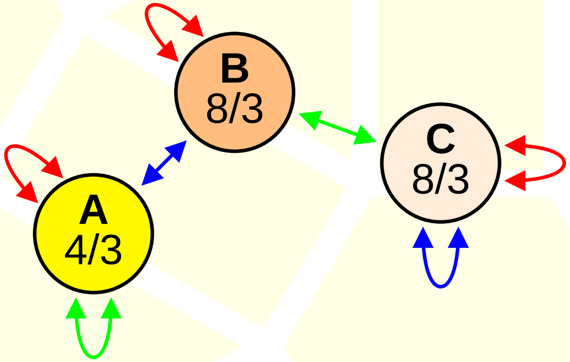


Formal conditions

For C a chamber and $i, j \in \{0, \dots, d\}$, we always have

- $s_i(s_i(C)) = C$
- $s_i(s_j(s_i(s_j(C)))) = C$ if $|i - j| > 1$

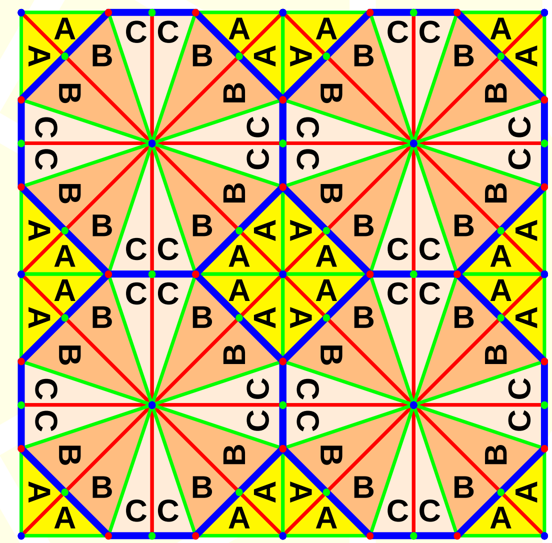


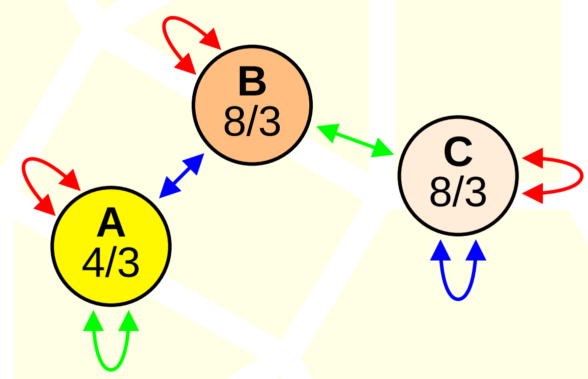


Formal conditions

For C a chamber and $i, j \in \{0, \ldots, d\}$, we always have

- $s_i(s_i(C)) = C$
- $s_i(s_j(s_i(s_j(C)))) = C$ if $|i - j| > 1$
- $(s_i \circ s_j)^{m_{i,j}(C)}(C) = C$

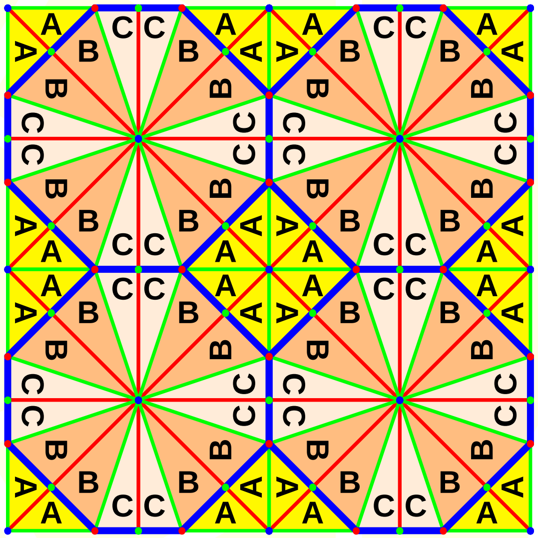


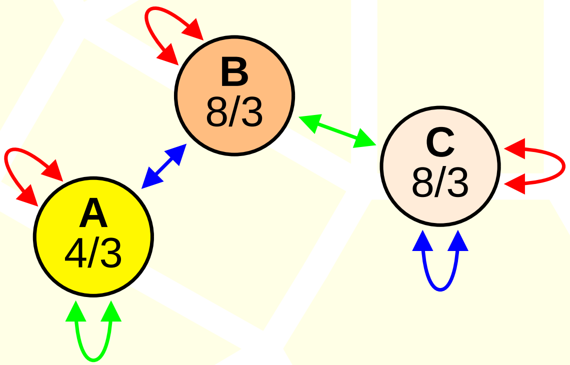


Formal conditions

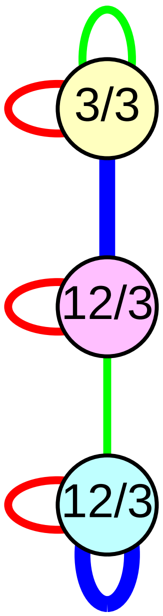
For C a chamber and $i, j \in \{0, \ldots, d\}$, we always have

- $s_i(s_i(C)) = C$
- $s_i(s_j(s_i(s_j(C)))) = C$ if $|i - j| > 1$
- $(s_i \circ s_j)^{m_{i,j}(C)}(C) = C$
- $m_{i,j}(C) = m_{i,j}(s_i(C)) = m_{i,j}(s_j(C))$





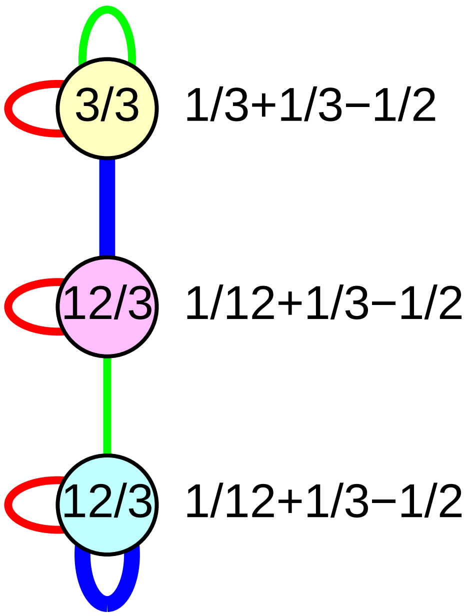
The curvature test



For each chamber C, compute

$$\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{2}.$$

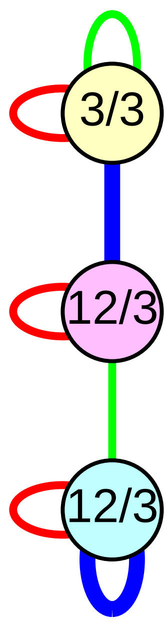
The curvature test



For each chamber C, compute

$$\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{2}.$$

The curvature test



1/3+1/3−1/2 = 1/6

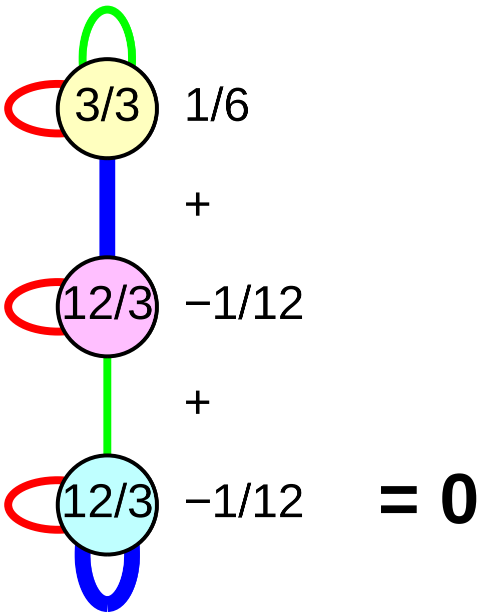
1/12+1/3−1/2 = −1/12

1/12+1/3−1/2 = −1/12

For each chamber C, compute

$$\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{2}.$$

The curvature test

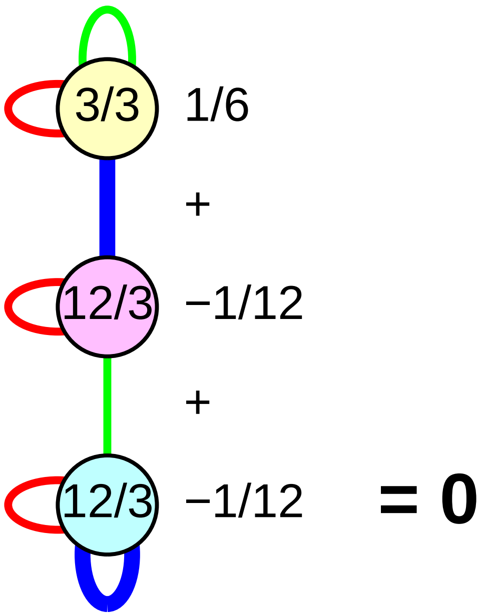


For each chamber C, compute

$$\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{2}.$$

Sum up to obtain the curvature K.

The curvature test



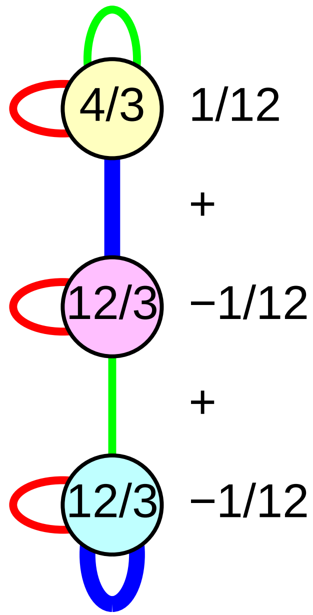
For each chamber C, compute

$$\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{2}.$$

Sum up to obtain the curvature K.

$K = 0 \quad \Rightarrow$ tiling of the euclidean plane.

The curvature test



= −1/12

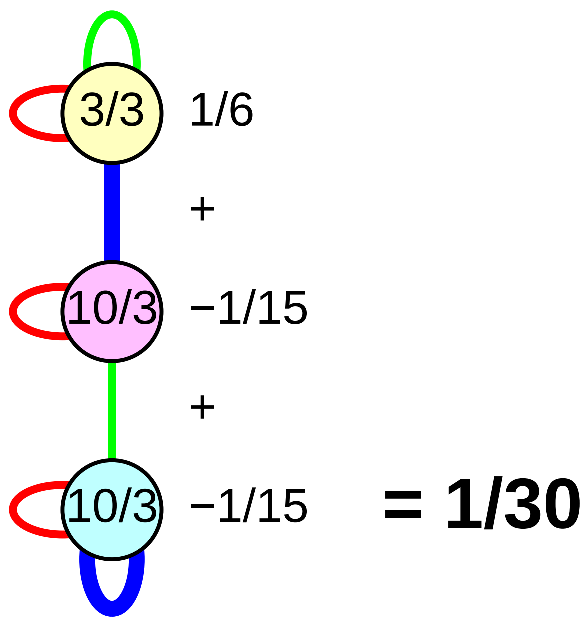
For each chamber C, compute

$$\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{2}.$$

Sum up to obtain the curvature K.

$K = 0 \quad \Rightarrow$ tiling of the euclidean plane.
$K < 0 \quad \Rightarrow$ tiling of the hyperbolic plane.

The curvature test



For each chamber C, compute
$$\frac{1}{m_{01}(C)} + \frac{1}{m_{12}(C)} - \frac{1}{2}.$$

Sum up to obtain the curvature K.

$K = 0 \quad \Rightarrow$ tiling of the euclidean plane.

$K < 0 \quad \Rightarrow$ tiling of the hyperbolic plane.

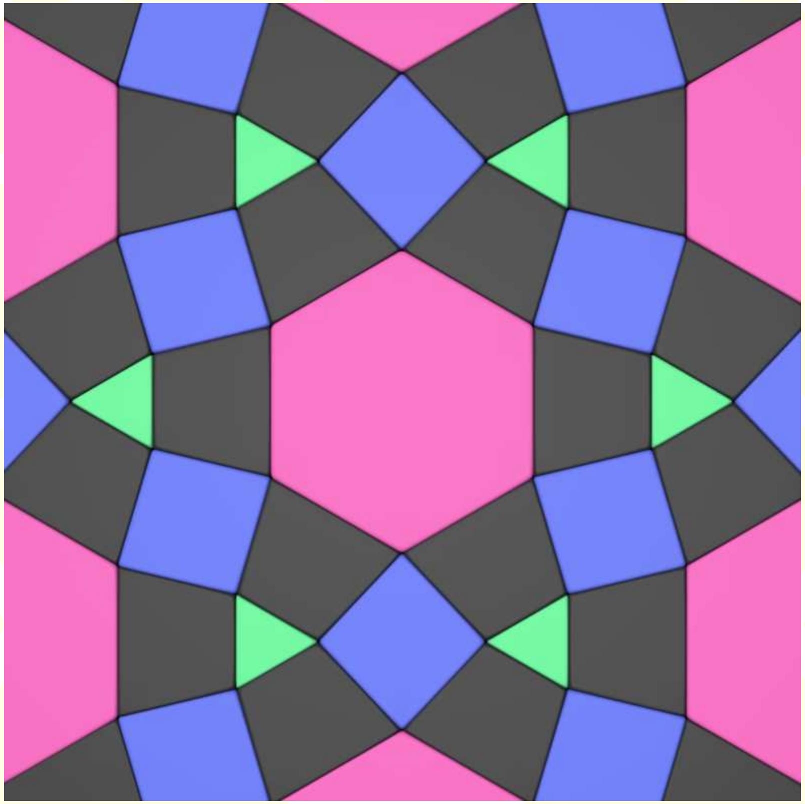
$K > 0 \quad \Rightarrow$ possibly a tiling of the sphere.

Why D-symbols?

- Easy to represent on a computer.
- If two tilings have the same D-symbol, they are equivariantly equivalent.
- Many properties of tilings can be easily translated into the language of D-symbols.

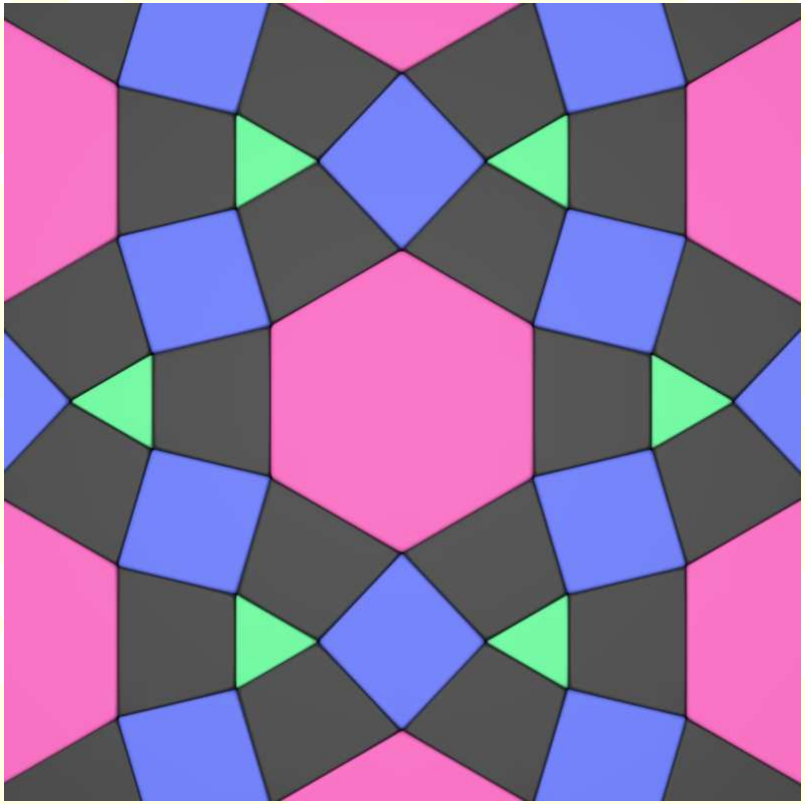
$\Rightarrow$ D-symbols are great for encoding tilings and for solving classification problems.

Example: Heaven & Hell tilings



- Each edge separates one black and one non-black tile.
- All black tiles are related by symmetry.

Example: Heaven & Hell tilings

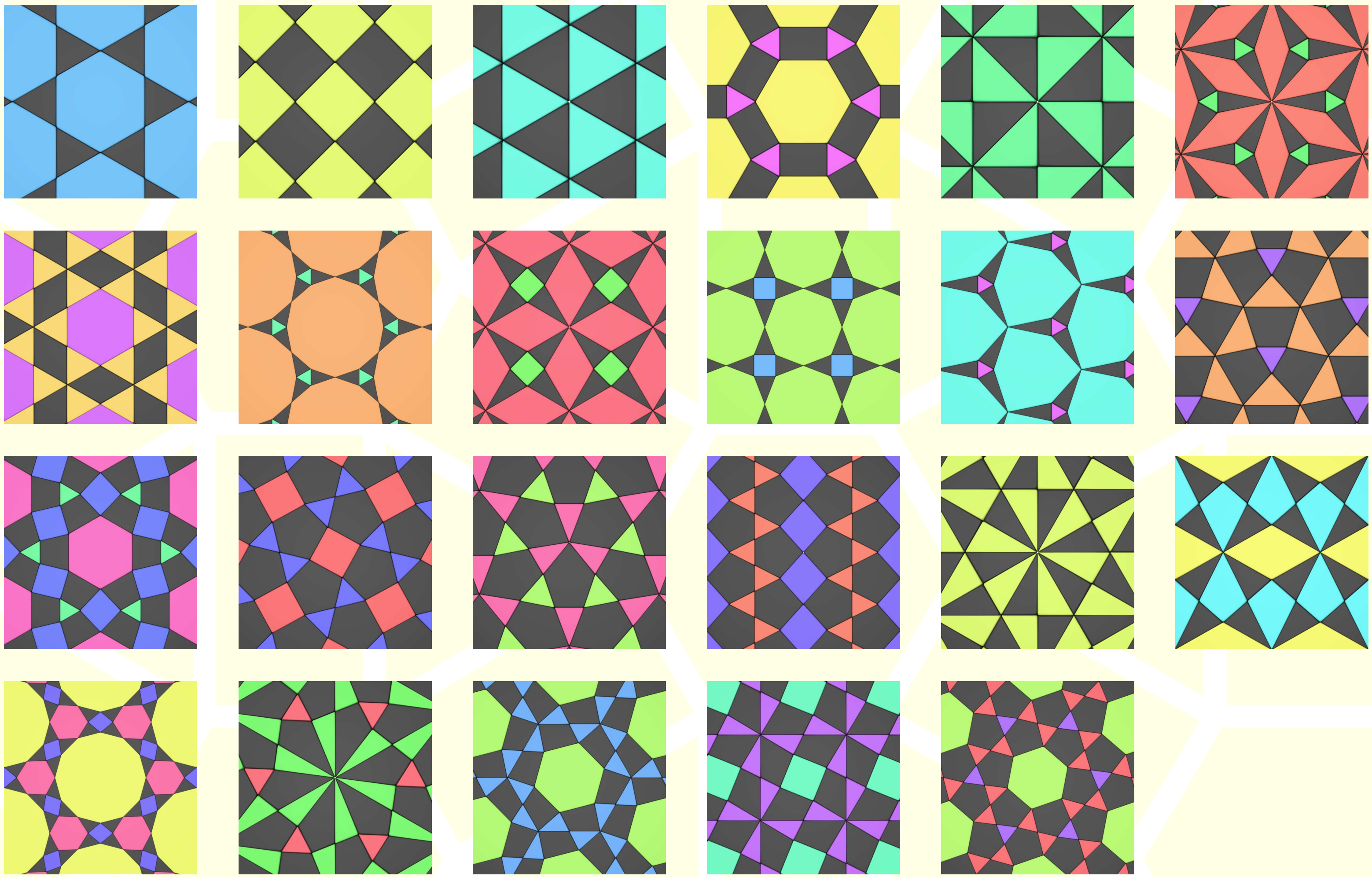


- Each edge separates one black and one non-black tile.
- All black tiles are related by symmetry.

There are 23 topological types of such tilings on the ordinary plane.

(A.W.M. DRESS, D.H. HUSON. *Revue Topologie Structurale,* 1991)

All heaven and hell



Transitivity

A tiling is called

- vertex-p-transitive if there are p kinds of vertex up to symmetry.

Transitivity

A tiling is called

- vertex-p-transitive if there are p kinds of vertex up to symmetry.
- edge-q-transitive if there are q kinds of edge up to symmetry.

Transitivity

A tiling is called

- vertex-p-transitive if there are p kinds of vertex up to symmetry.
- edge-q-transitive if there are q kinds of edge up to symmetry.
- etc.

Transitivity

A tiling is called

- vertex-p-transitive if there are p kinds of vertex up to symmetry.
- edge-q-transitive if there are q kinds of edge up to symmetry.
- etc.

It is convenient to combine those numbers in a transitivity symbol pqr or - for 3d tilings - $pqrs$.

Transitivity

A tiling is called

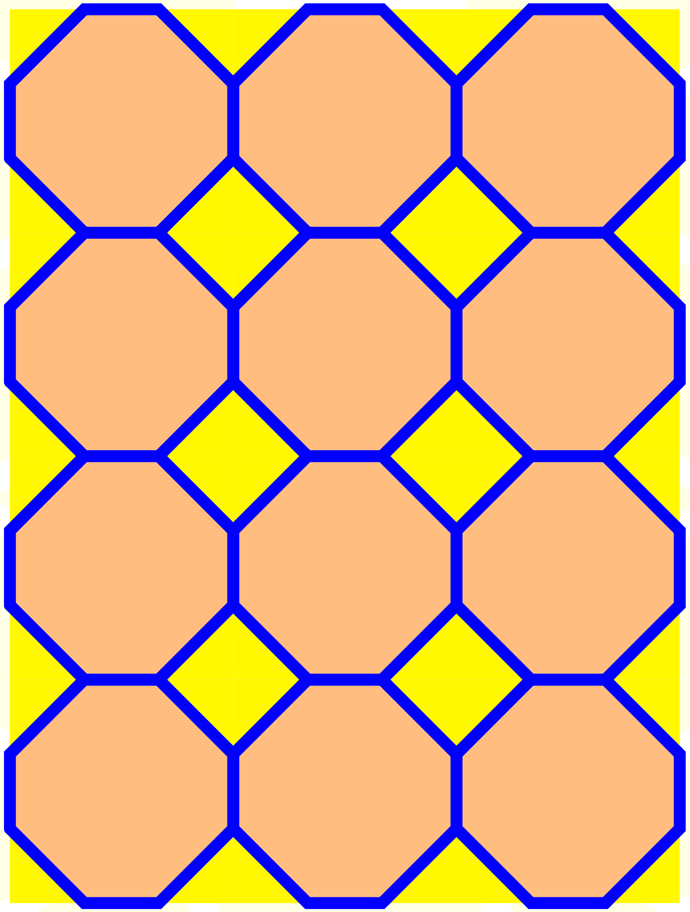
- vertex-p-transitive if there are p kinds of vertex up to symmetry.
- edge-q-transitive if there are q kinds of edge up to symmetry.
- etc.

It is convenient to combine those numbers in a transitivity symbol pqr or - for 3d tilings - $pqrs$.

Tile-1-transitive tilings are sometimes called isohedral.

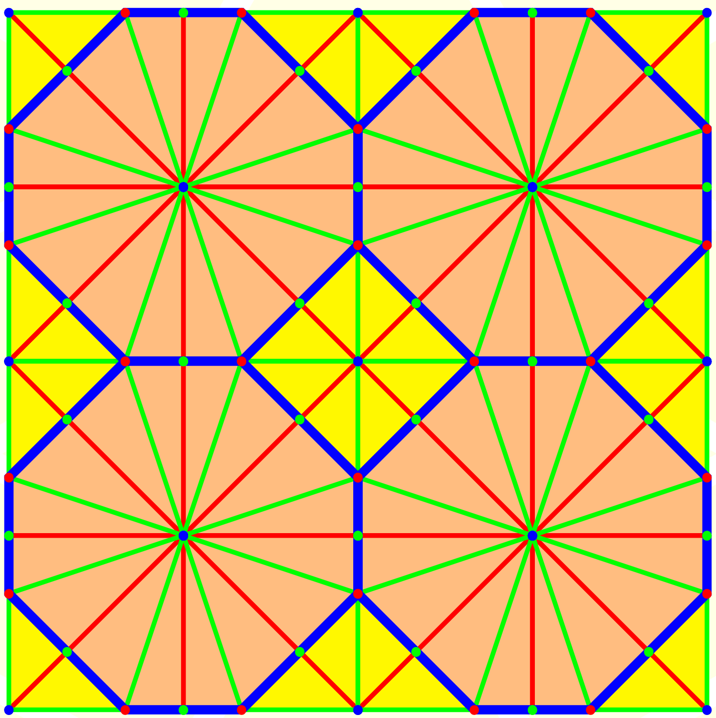
Quick test

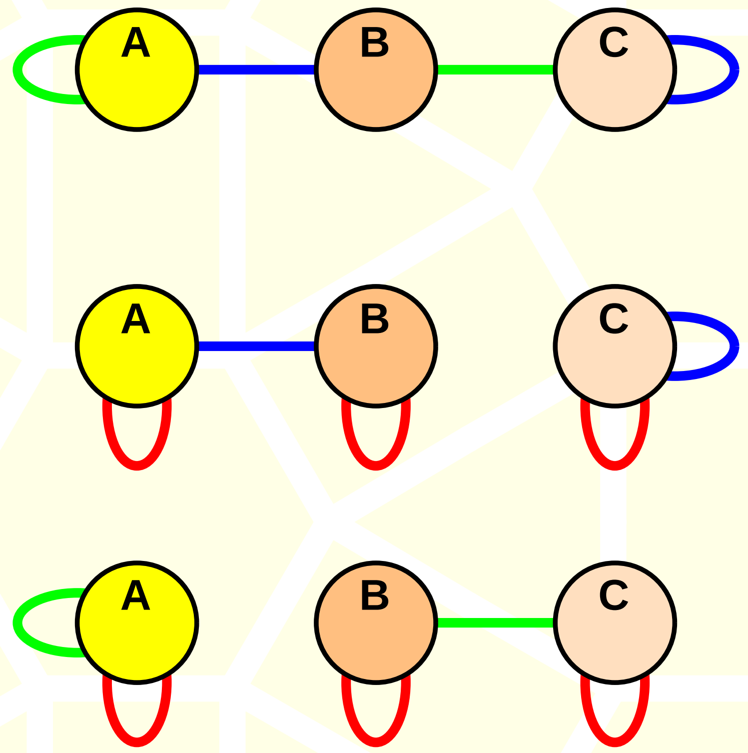
What is the transitivity of this tiling?



Quick test

What is the transitivity of this tiling?





It is 122.

Fundamental tilings

- A tiling is called fundamental if it is tile-1-transitive and the site symmetry for each tile is trivial.

Fundamental tilings

- A tiling is called fundamental if it is tile-1-transitive and the site symmetry for each tile is trivial.
- The tiles are then asymmetric units (a.k.a. fundamental domains) for the tiling's symmetry group.

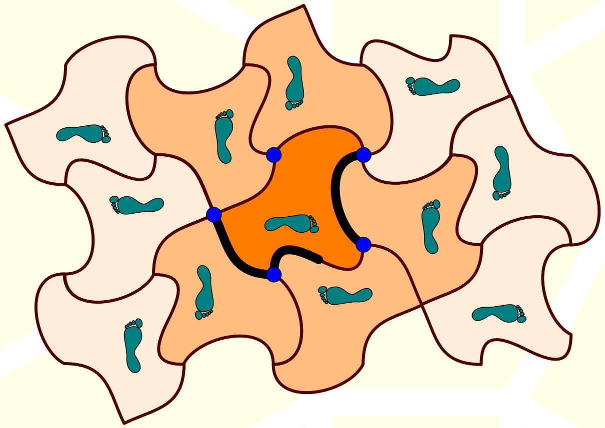
Fundamental tilings

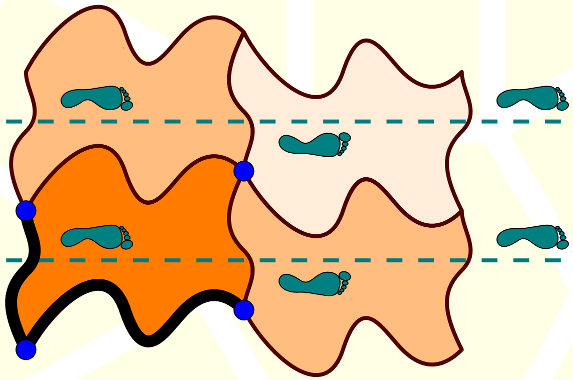
- A tiling is called fundamental if it is tile-1-transitive and the site symmetry for each tile is trivial.
- The tiles are then asymmetric units (a.k.a. fundamental domains) for the tiling's symmetry group.
- Heesch proved (around 1935) that there are exactly 46 equivariant types of fundamental tilings in the plane.

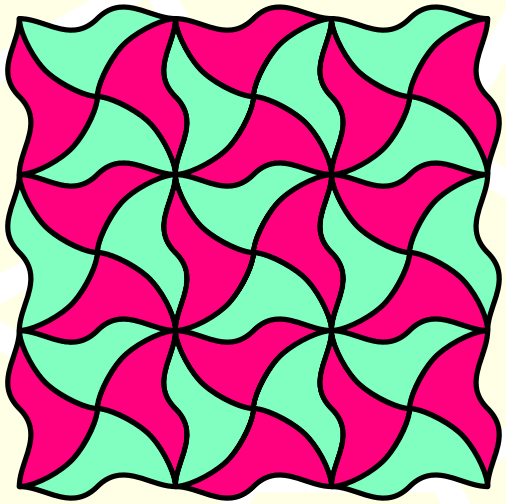
Fundamental tilings

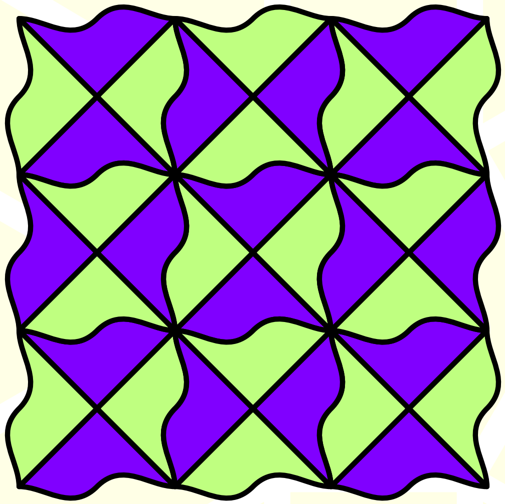
- A tiling is called fundamental if it is tile-1-transitive and the site symmetry for each tile is trivial.
- The tiles are then asymmetric units (a.k.a. fundamental domains) for the tiling's symmetry group.
- Heesch proved (around 1935) that there are exactly 46 equivariant types of fundamental tilings in the plane.
- These fall into 11 topological types which are known as Laves nets.

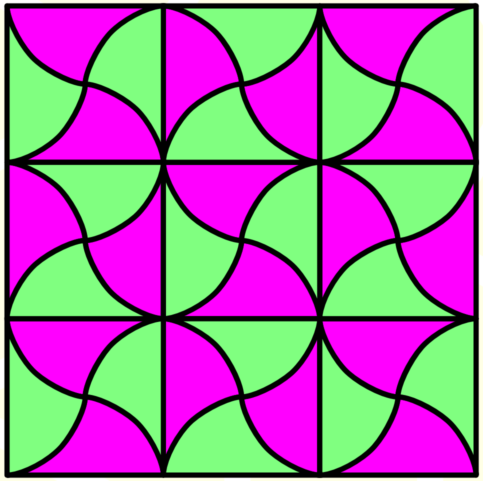
Some of Heesch's tilings

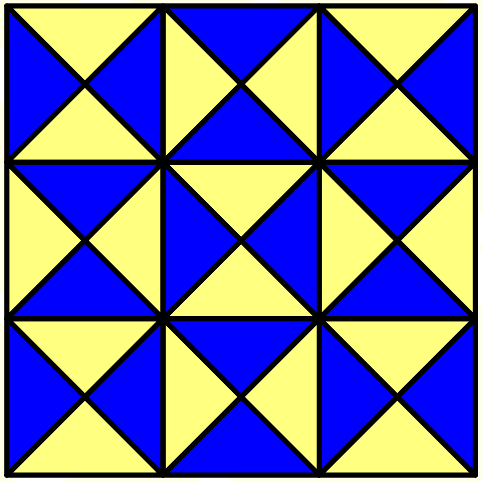












A hierarchy of tilings

- A tile with trivial site symmetry can be split to increase the tile-transitivity of a tiling.

A hierarchy of tilings

- A tile with trivial site symmetry can be split to increase the tile-transitivity of a tiling.
- Equivalent tiles surrounding a center of symmetry can be glued together to form a tile with non-trivial site symmetry.

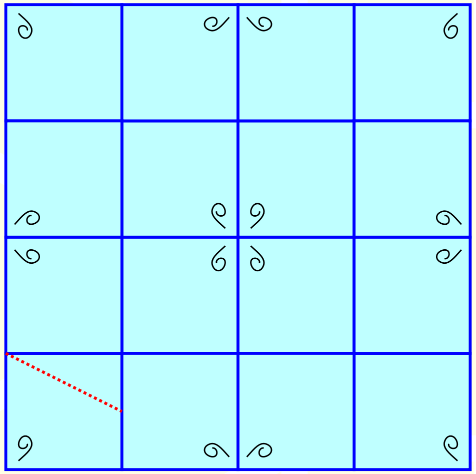
A hierarchy of tilings

- A tile with trivial site symmetry can be split to increase the tile-transitivity of a tiling.
- Equivalent tiles surrounding a center of symmetry can be glued together to form a tile with non-trivial site symmetry.
- Repeated splitting and glueing, starting from the fundamental tilings, produces all tilings of the plane.

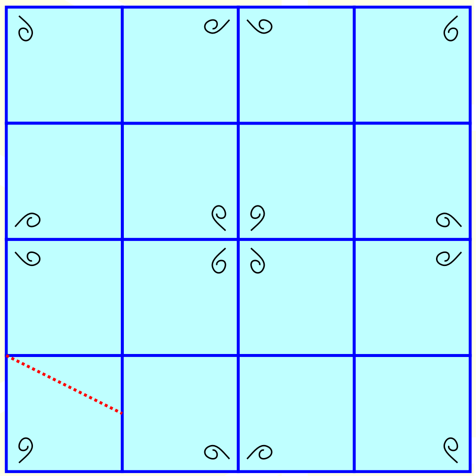
A hierarchy of tilings

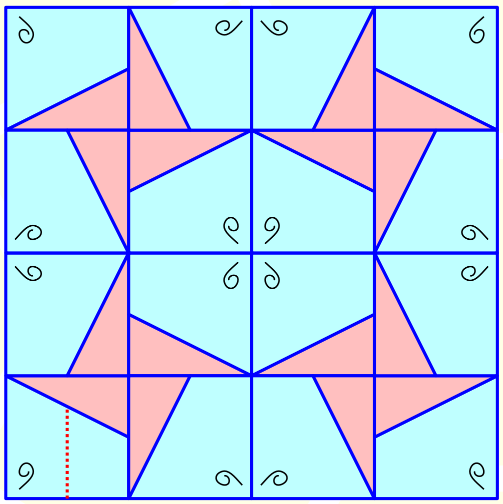
- A tile with trivial site symmetry can be split to increase the tile-transitivity of a tiling.
- Equivalent tiles surrounding a center of symmetry can be glued together to form a tile with non-trivial site symmetry.
- Repeated splitting and glueing, starting from the fundamental tilings, produces all tilings of the plane.
- The same process works for tilings of the sphere and the hyperbolic plane.

Split and glue

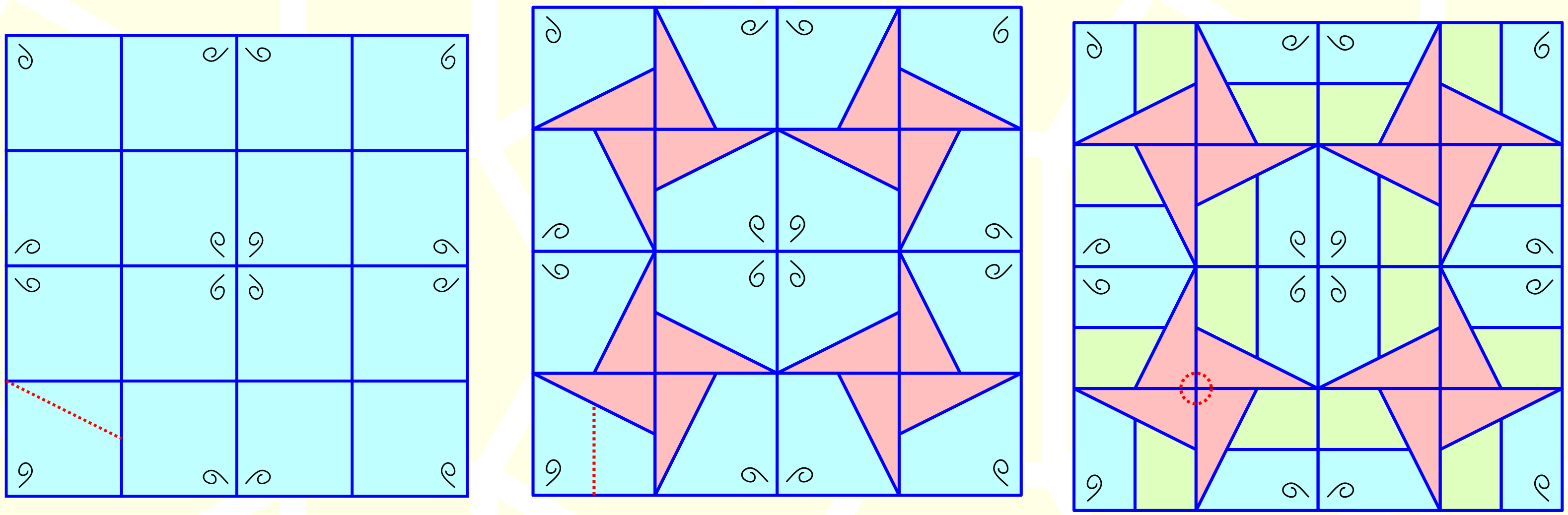


Split and glue

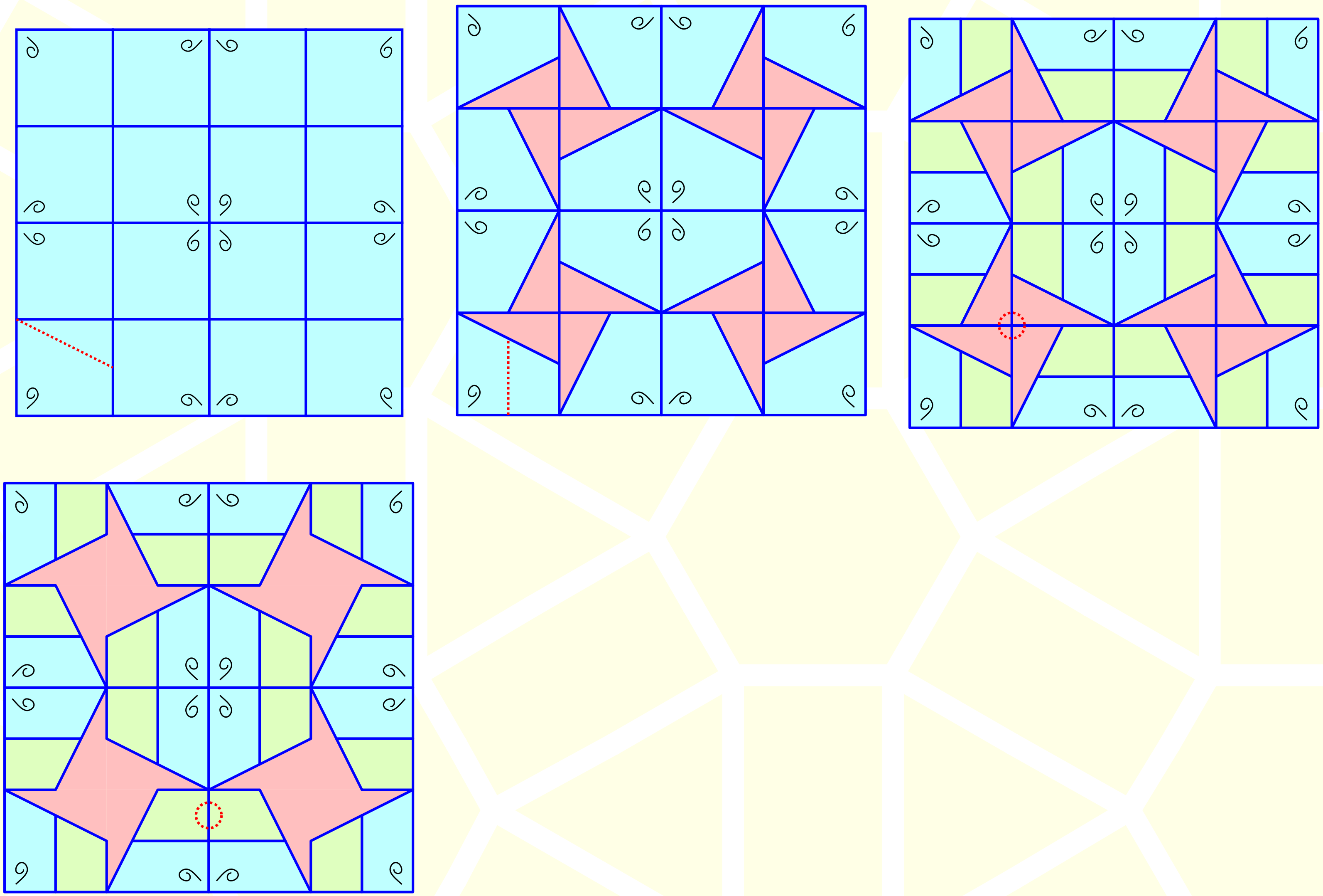




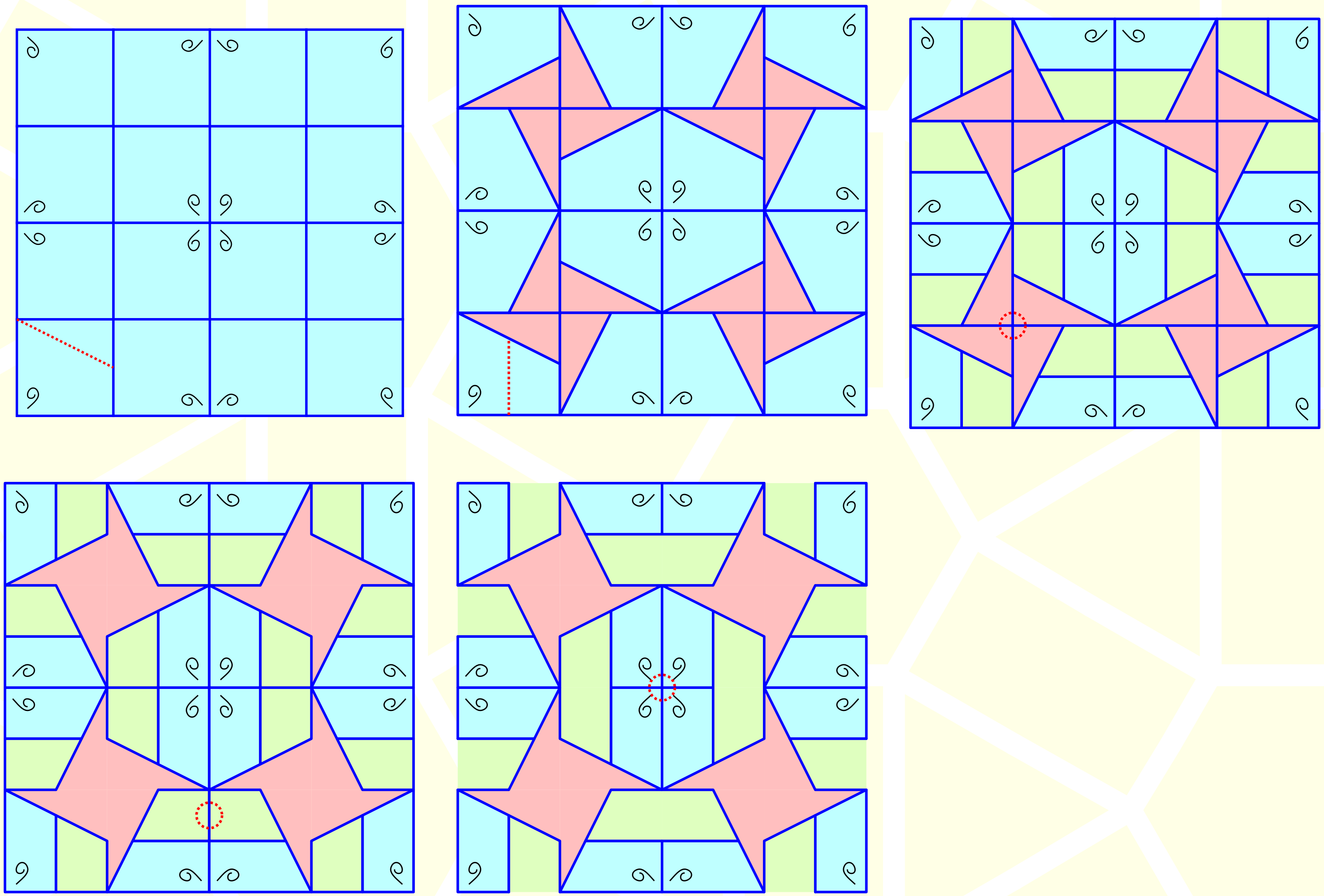
Split and glue



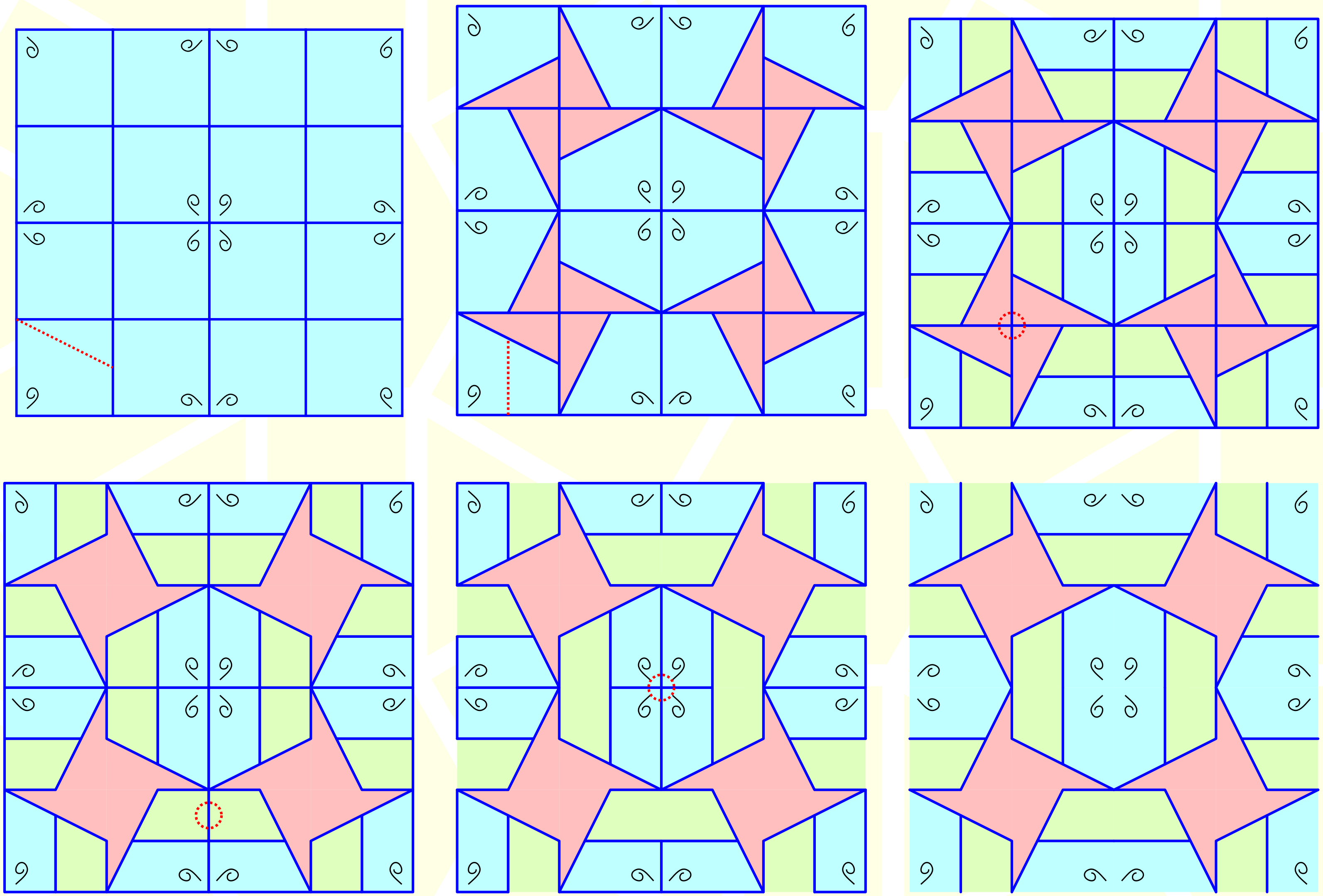
Split and glue



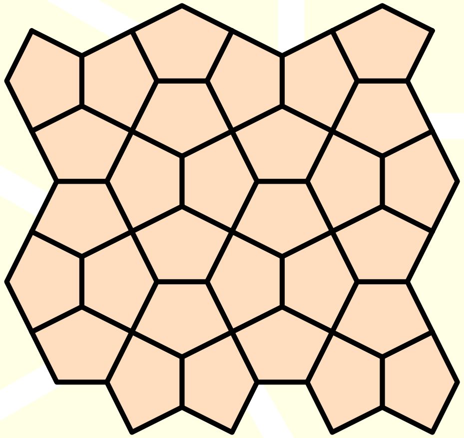
Split and glue



Split and glue

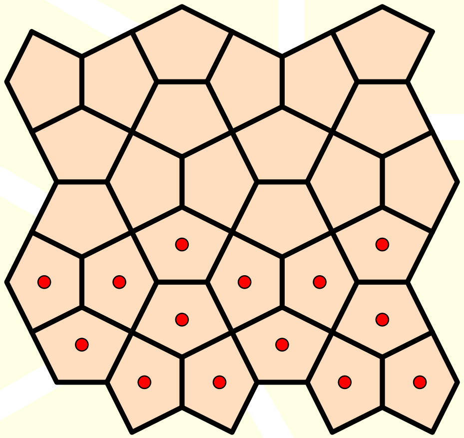


Dualization



To dualize a tiling:

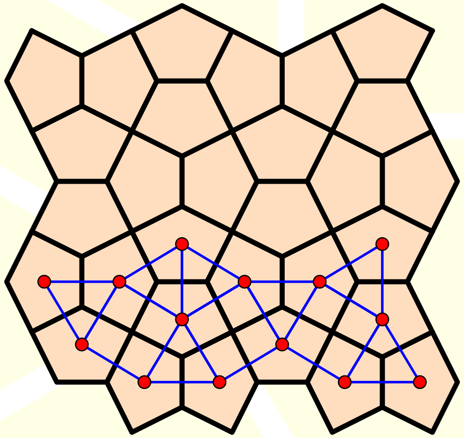
Dualization



To dualize a tiling:

- Put a new vertex in each tile.

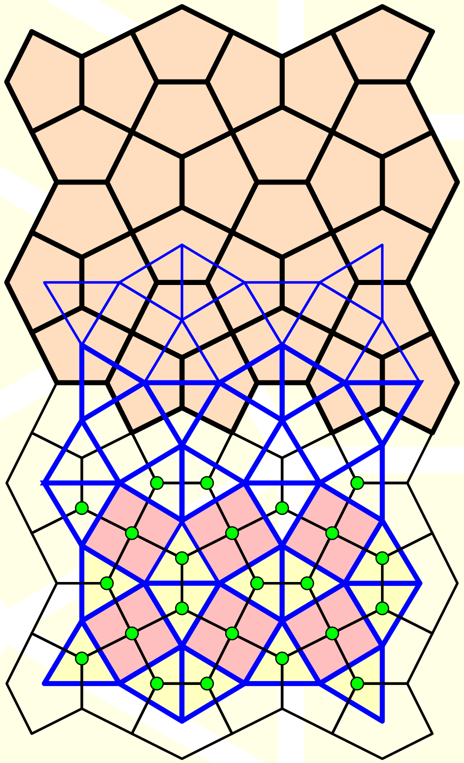
Dualization



To dualize a tiling:

- Put a new vertex in each tile.
- Connect new vertices in adjacent tiles.

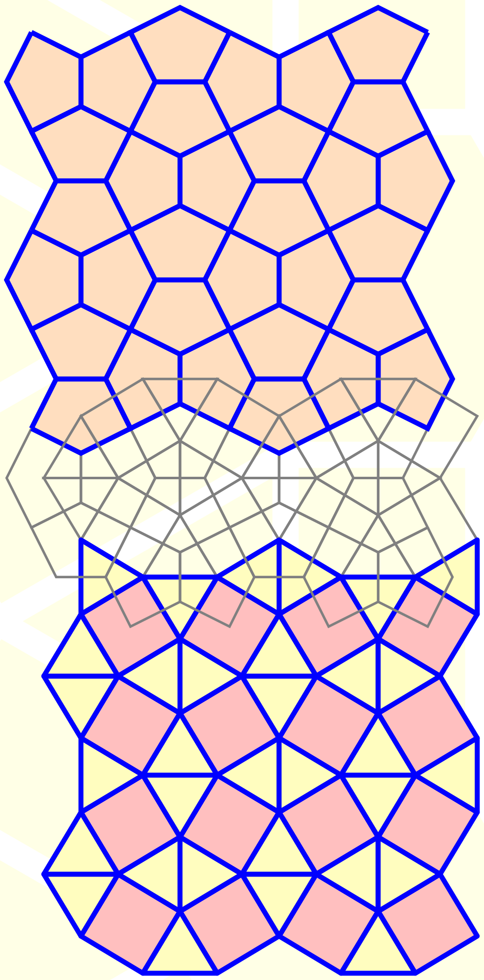
Dualization



To dualize a tiling:

- Put a new vertex in each tile.
- Connect new vertices in adjacent tiles.
- There is then a new tile for each old vertex.

Dualization

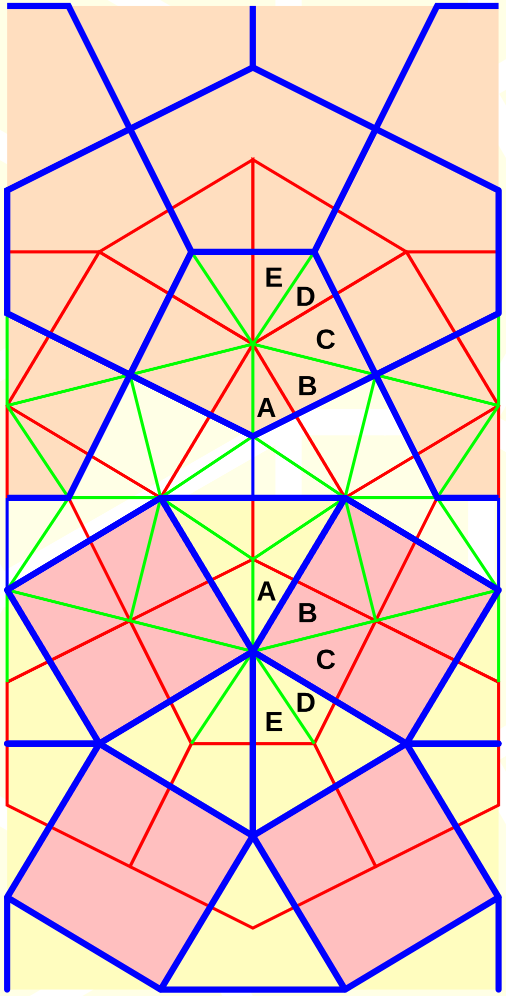


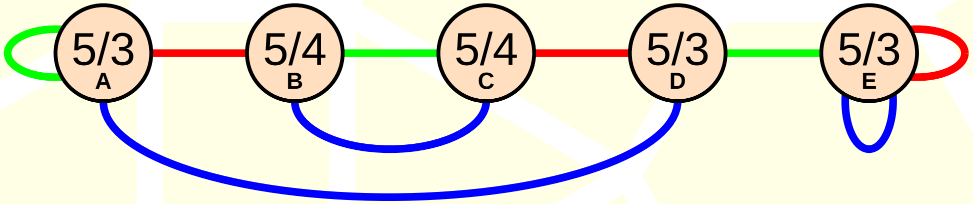
To dualize a tiling:

- Put a new vertex in each tile.
- Connect new vertices in adjacent tiles.
- There is then a new tile for each old vertex.

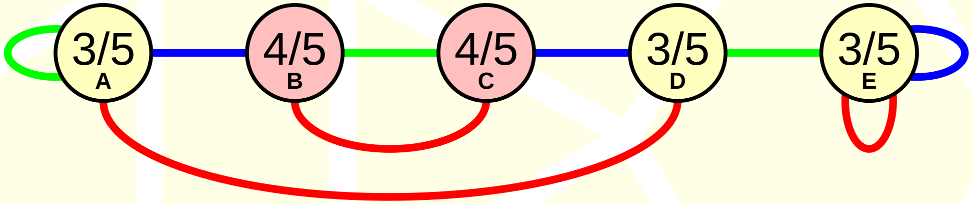
In fact, the dual of the dual is the original tiling.

Dual D-symbols



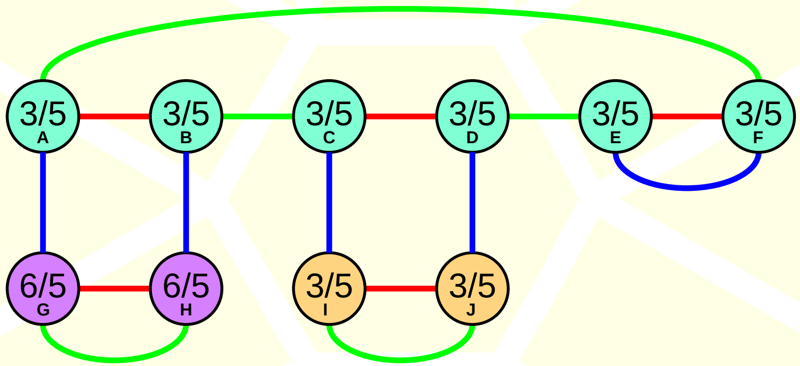


Dualization (in 2d) switches s_0 with s_2 (red with blue) and m_{01} with m_{12}.

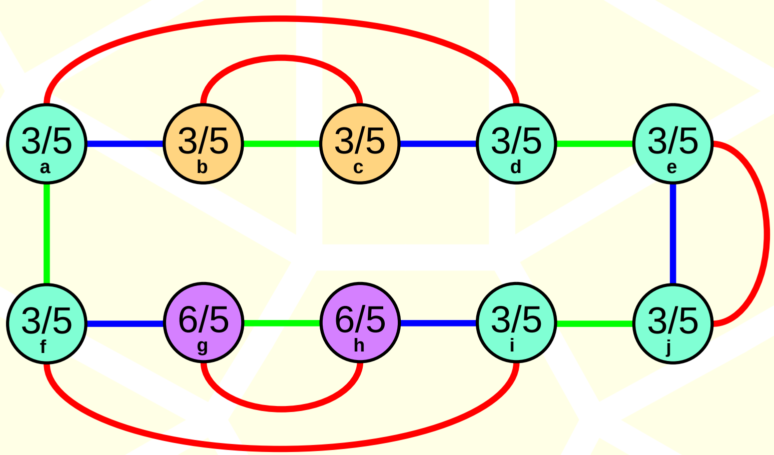


Comparing symbols

Are these two symbols really the same?

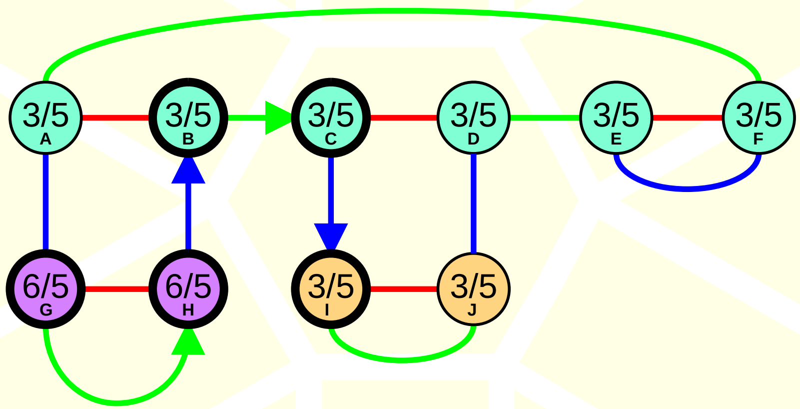


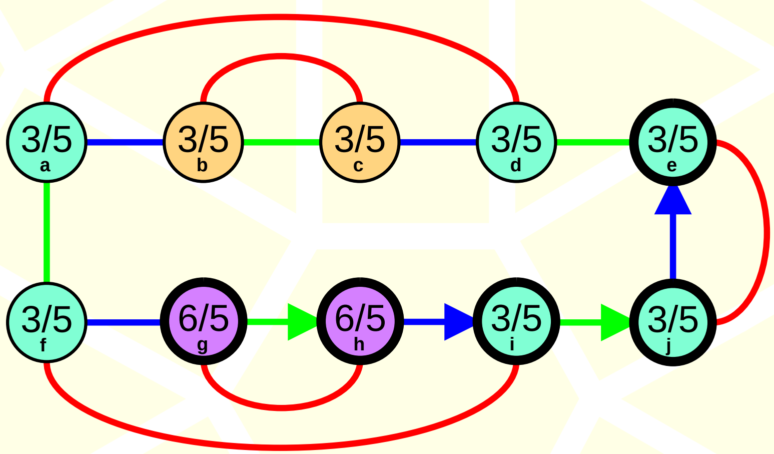
- G matches g or h.



Comparing symbols

Are these two symbols really the same?

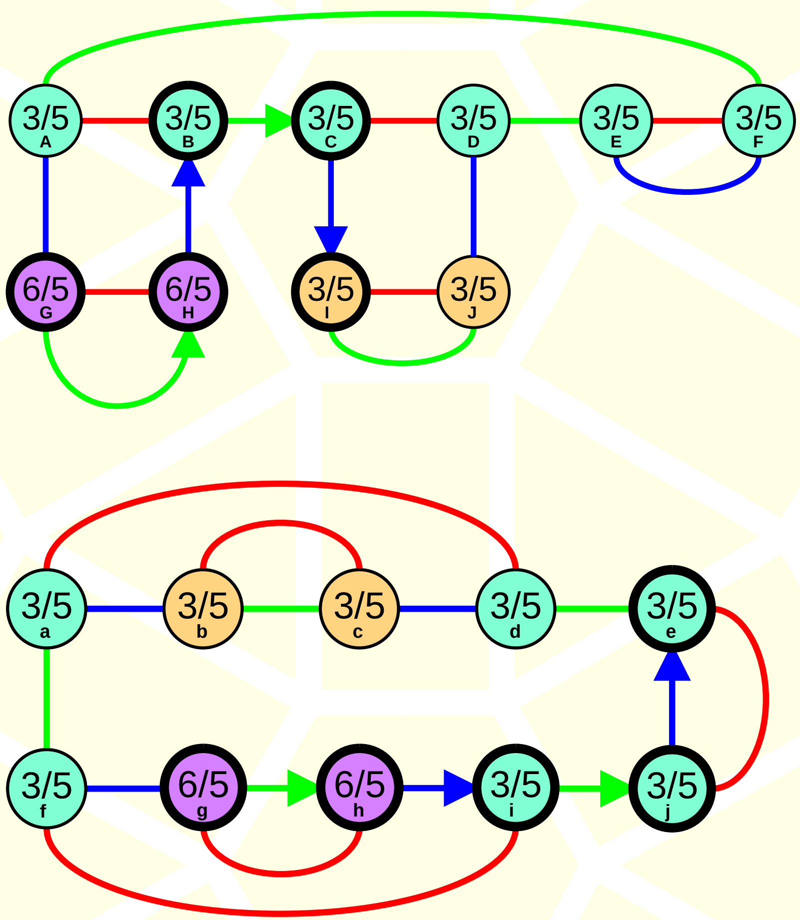




- G matches g or h.
- Tracing green and blue edges: G,g → H,h → B,i → C,j → I,e.

Comparing symbols

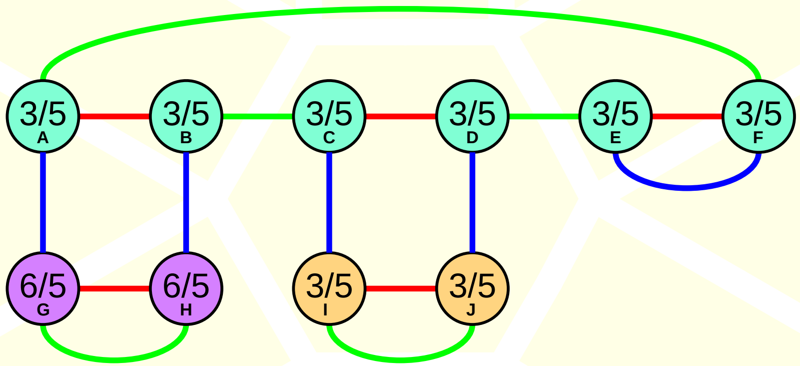
Are these two symbols really the same?

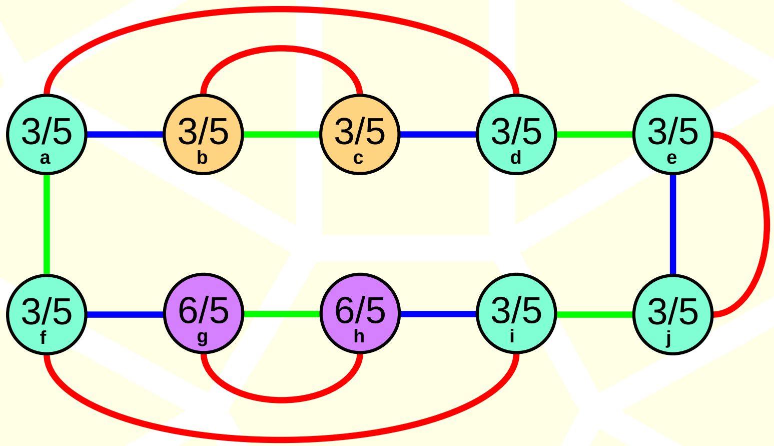


- G matches g or h.
- Tracing green and blue edges: G,g $\longrightarrow$ H,h $\longrightarrow$ B,i $\longrightarrow$ C,j $\longrightarrow$ I,e.
- But j and e also share a red edge; C and I don't.

Comparing symbols

Are these two symbols really the same?

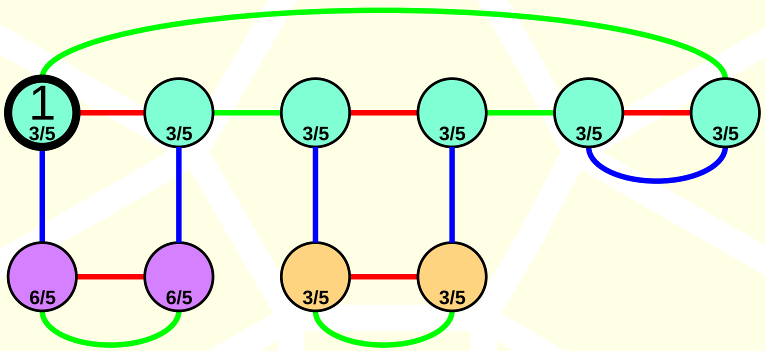




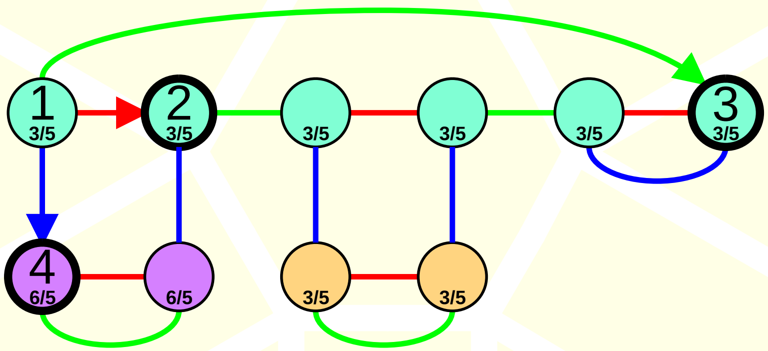
- G matches g or h.
- Tracing green and blue edges: G,g $\longrightarrow$ H,h $\longrightarrow$ B,i $\longrightarrow$ C,j $\longrightarrow$ I,e.
- But j and e also share a red edge; C and I don’t.
- G,h works.

Traversing D-symbols

- Pick a start vertex.

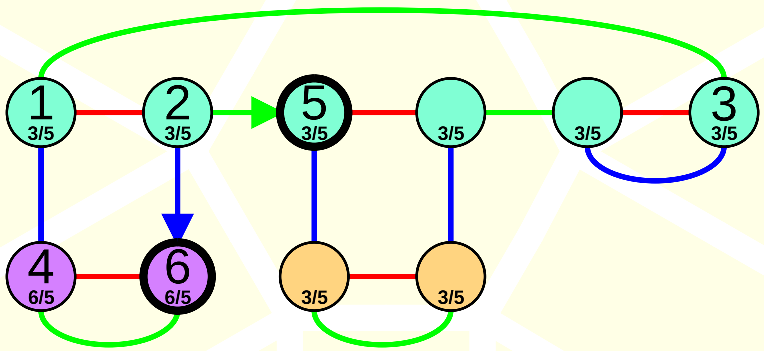


Traversing D-symbols



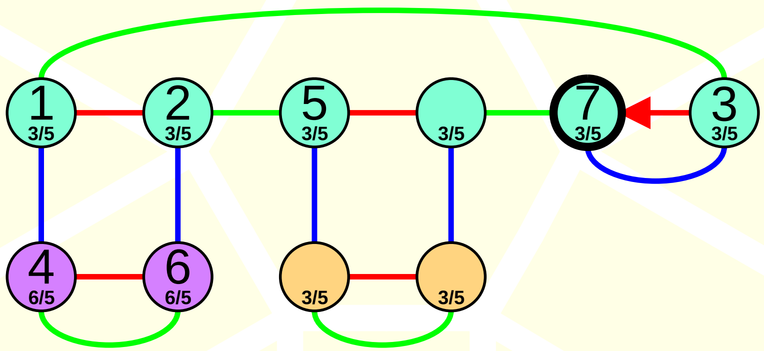
- Pick a start vertex.
- Visit its red, green and blue neighbor in order.

Traversing D-symbols



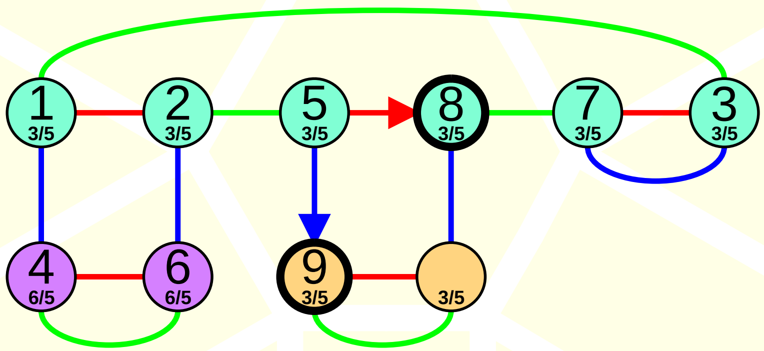
- Pick a start vertex.
- Visit its red, green and blue neighbor in order.
- Repeat in the order the vertexes were visited.

Traversing D-symbols



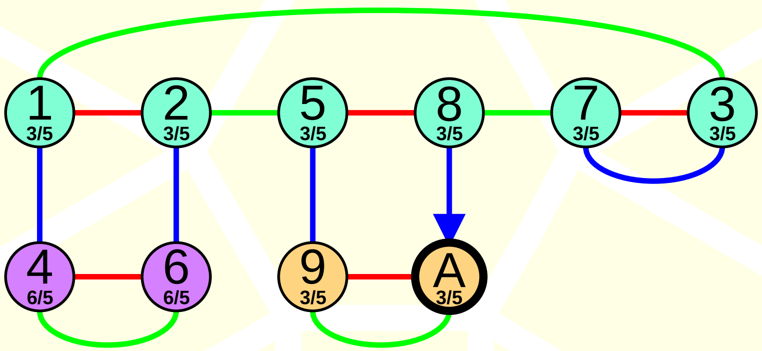
- Pick a start vertex.
- Visit its red, green and blue neighbor in order.
- Repeat in the order the vertexes were visited.

Traversing D-symbols



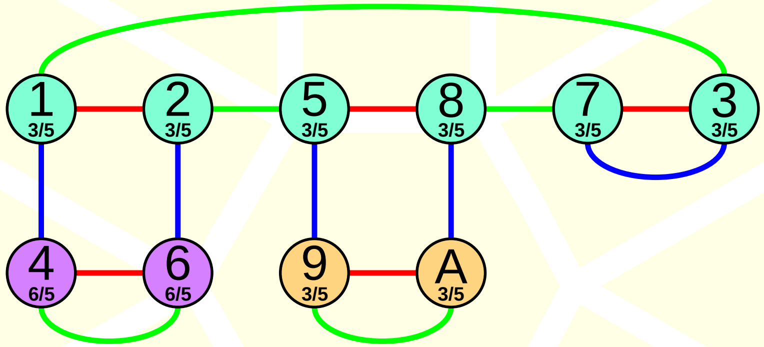
- Pick a start vertex.
- Visit its red, green and blue neighbor in order.
- Repeat in the order the vertexes were visited.

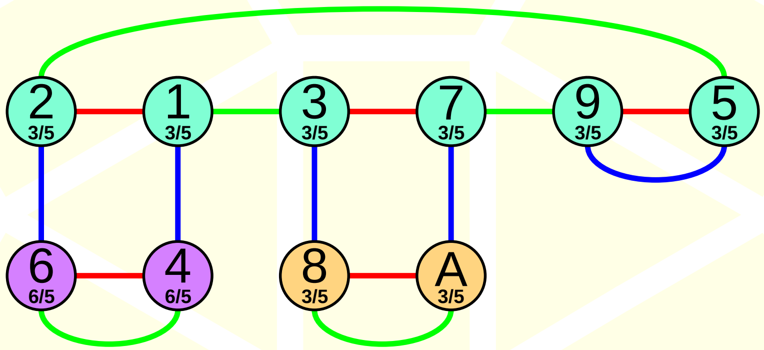
Traversing D-symbols



- Pick a start vertex.
- Visit its red, green and blue neighbor in order.
- Repeat in the order the vertexes were visited.

Traversing D-symbols

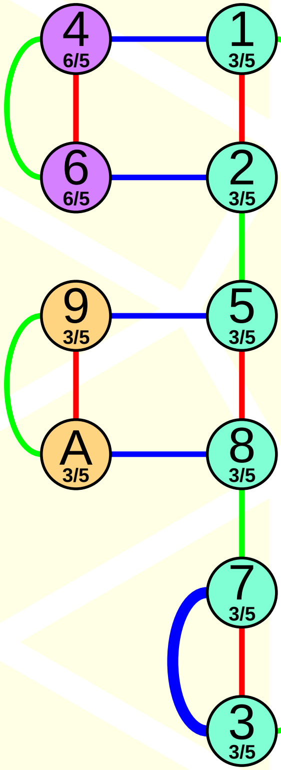




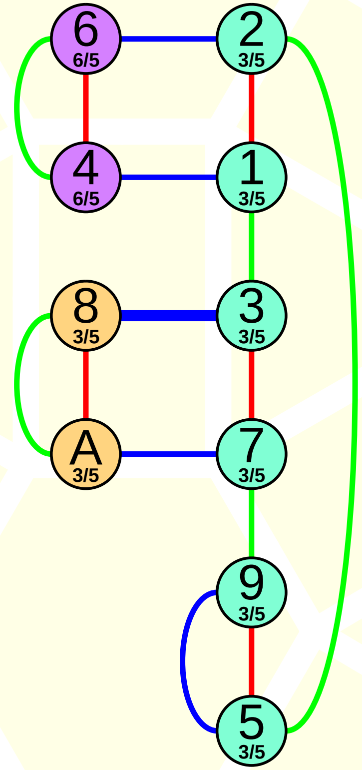
- Pick a start vertex.
- Visit its red, green and blue neighbor in order.
- Repeat in the order the vertexes were visited.

This produces a unique ordering for each choice of start vertex.

Using traversals

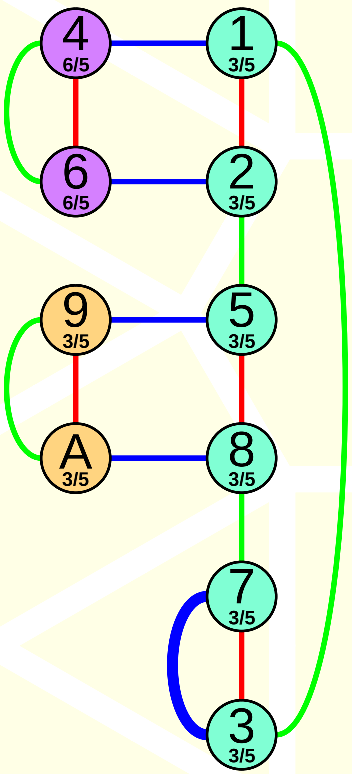


1: 2,3,4;3,5
2: 1,5,6;3,5
3: 7,1,**7**;3,5
4: 6,6,1;6,5
5: 8,2,9;3,5
6: 4,4,2;6,5
7: 3,8,3;3,5
8: 5,7,A;3,5
9: A,A,5;3,5
A: 9,9,8;3,5

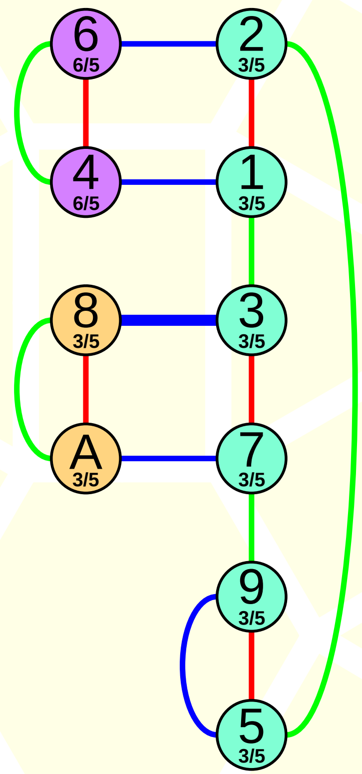


1: 2,3,4;3,5
2: 1,5,6;3,5
3: 7,1,**8**;3,5
4: 6,6,1;6,5
5: 9,2,9;3,5
6: 4,4,2;6,5
7: 3,9,A;3,5
8: A,A,3;3,5
9: 5,7,5;3,5
A: 8,8,7;3,5

Using traversals



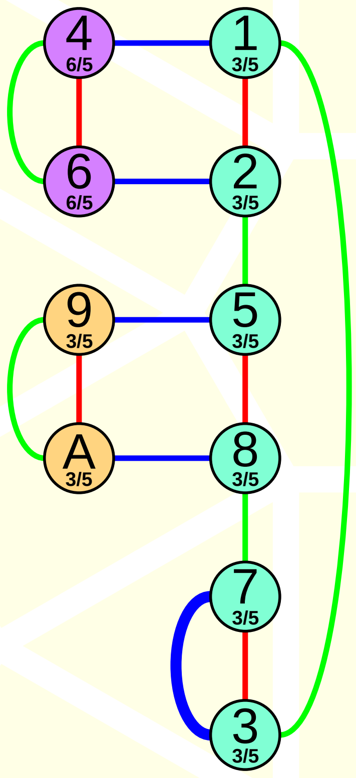
1: 2,3,4;3,5
2: 1,5,6;3,5
3: 7,1,**7**;3,5
4: 6,6,1;6,5
5: 8,2,9;3,5
6: 4,4,2;6,5
7: 3,8,3;3,5
8: 5,7,A;3,5
9: A,A,5;3,5
A: 9,9,8;3,5



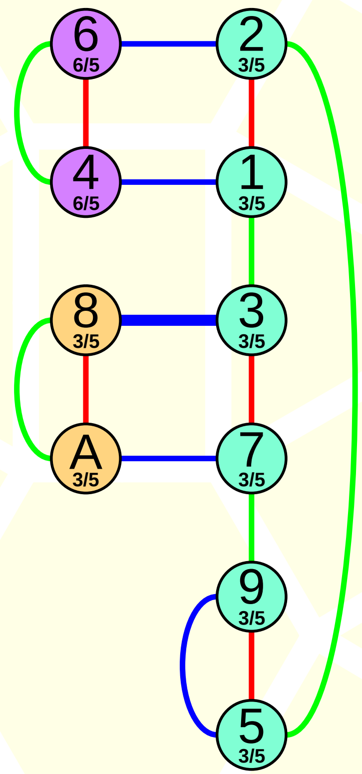
1: 2,3,4;3,5
2: 1,5,6;3,5
3: 7,1,**8**;3,5
4: 6,6,1;6,5
5: 9,2,9;3,5
6: 4,4,2;6,5
7: 3,9,A;3,5
8: A,A,3;3,5
9: 5,7,5;3,5
A: 8,8,7;3,5

Write down the s and m values for each vertex and compare by the first position that differs.

Using traversals



1: 2,3,4;3,5
2: 1,5,6;3,5
3: 7,1,**7**;3,5
4: 6,6,1;6,5
5: 8,2,9;3,5
6: 4,4,2;6,5
7: 3,8,3;3,5
8: 5,7,A;3,5
9: A,A,5;3,5
A: 9,9,8;3,5



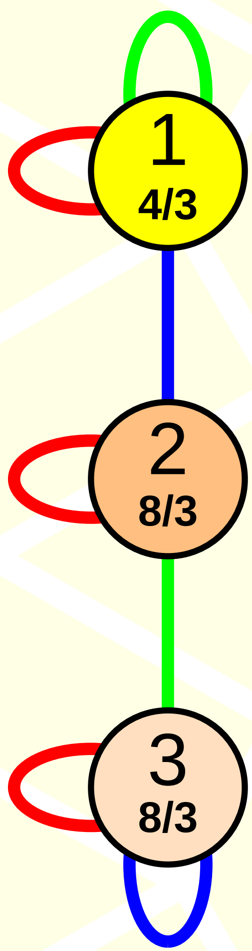
1: 2,3,4;3,5
2: 1,5,6;3,5
3: 7,1,**8**;3,5
4: 6,6,1;6,5
5: 9,2,9;3,5
6: 4,4,2;6,5
7: 3,9,A;3,5
8: A,A,3;3,5
9: 5,7,5;3,5
A: 8,8,7;3,5

Write down the s and m values for each vertex and compare by the first position that differs.

$\Rightarrow$ use best traversal as a canonical form.

The .ds file format

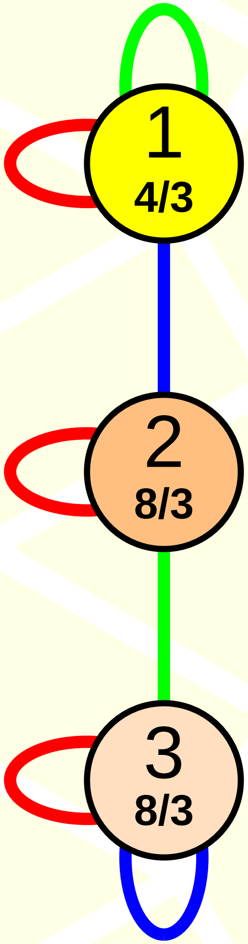
<1.1:



- Any pair of numbers.

The .ds file format

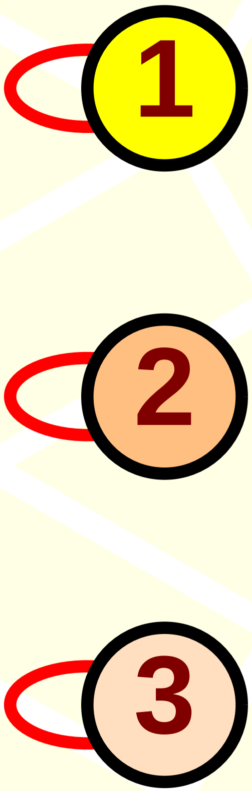
<1.1:3 2:



- Any pair of numbers.
- The size and dimension.

The .ds file format

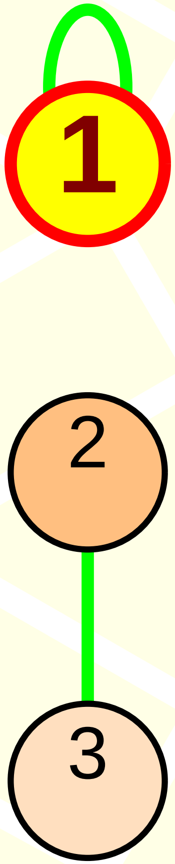
```
<1.1:3 2:1 2 3,
```



- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.

The .ds file format

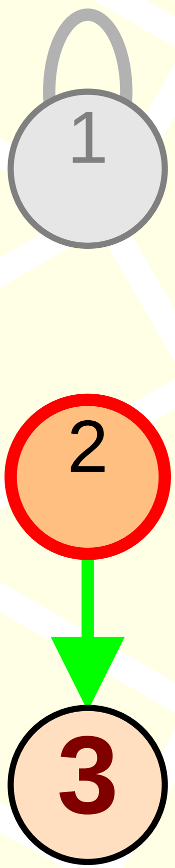
```
<1.1:3 2:1 2 3,1
```



- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.

The .ds file format

`<1.1:3 2:1 2 3,1 3,`



- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.

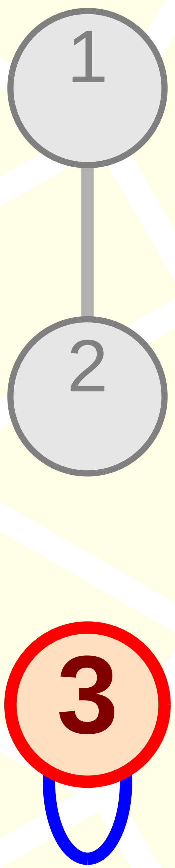
The .ds file format

<1.1:3 2:1 2 3,1 3,2

- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.

The .ds file format

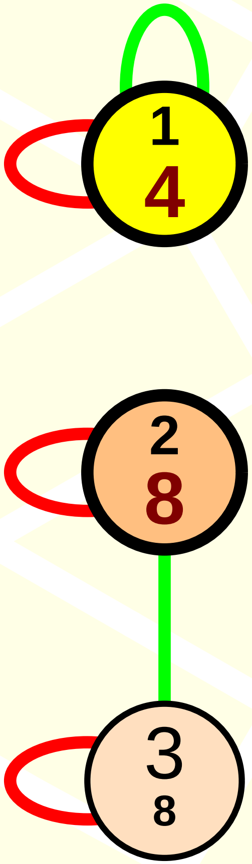
```
<1.1:3 2:1 2 3,1 3,2 3:
```



- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.

The .ds file format

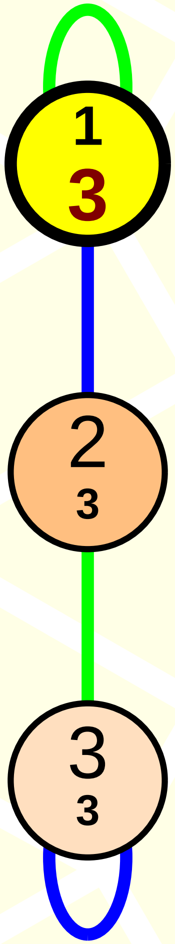
```
<1.1:3 2:1 2 3,1 3,2 3:4 8,
```



- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.
- Non-induced values of m_{01}, m_{12} in order.

The .ds file format

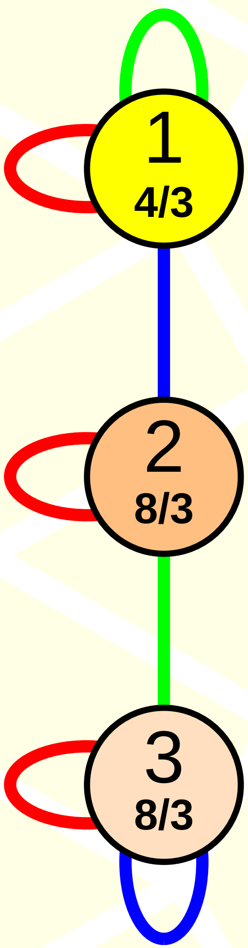
```
<1.1:3 2:1 2 3,1 3,2 3:4 8,3>
```



- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.
- Non-induced values of m_{01}, m_{12} in order.

The .ds file format

```
<1.1:3 2:1 2 3,1 3,2 3:4 8,3>
```



- Any pair of numbers.
- The size and dimension.
- Images of s_0, s_1, s_2 in order.
- Non-induced values of m_{01}, m_{12} in order.

One line per symbol - mind the punctuation.

Thanks for your attention!

Software is available at

www.gavrog.org